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"Positivity Conditions for Higgs Bundles and Applications"

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People who say “it cannot be done”
should not interrupt
those who are doing it.

Contents

Introduction

Motivations

The basic underlying idea of this thesis is to use *Higgs bundles* as a probe to study the geometry of smooth projective varieties, defined over an algebraically closed field of characteristic 0. The starting point of this approach may be found in the work of Simpson on the *non-Abelian Hodge theory* [61, 62]. In particular, in this thesis we shall use *notions of positivity for Higgs bundles* to improve some results of Yau ([69]), Miyaoka ([52]) and Simpson ([61]) about smooth complex projective varieties of general type (see Chapter 5).

These notions of positivity also relate to a conjecture about a class of Higgs bundles satisfying a "*strong semistability condition*" (the *curve semistability*, see Definition 4.1.1). The conjecture was formulated by Bruzzo and Graña Otero in 2006 and is still open, although some progress has recently been made.

This thesis also contains some contributions in that direction.

Positivity conditions for line bundles

Let us start from reviewing positivity conditions for line bundles.

Let X be a projective scheme over an algebraically closed field $\mathbb{K}$. A line bundle L over X is *very ample* (*relatively to* $\operatorname{Spec}(\mathbb{K})$) if there is a closed embedding $i\colon X \to \mathbb{P}^N_{\mathbb{K}}$ for some $N \in \mathbb{N}_{\geq 1}$ such that $i^*\mathcal{O}(1) = L$ ([30, Second Definition at page 120]; this definition is slightly different from the original one [27, Définition 4.4.2]). A line bundle L over X is *ample* if $L^{\otimes m}$ is very ample (relatively to $\operatorname{Spec}(\mathbb{K})$) for some $m \in \mathbb{N}_{\geq 1}$ (cfr. [30, Theorem II.7.6]).

To be clear, Grothendieck and Hartshorne have defined very ample and ample line bundles not only over projective schemes. By *Cartan, Serre, Grothendieck Theorem*, these definitions of ample line bundle (over projective schemes) given by Grothendieck and Hartshorne are equivalent (see [28, Proposition 2.6.1] and [30, Proposition III.5.3]).

In different papers, Nakai, Moišezon and Kleiman have proved that a line bundle L over X is ample if and only if for any positive-dimensional subvariety V of X the following inequality

$$\int_V c_1(L)^{\dim V} > 0$$

holds ([45, Theorem 1.2.23]).

"Passing to the limit"[1], that is considering the line bundles L over X such that for any positive-dimensional subvariety V of X the following inequality

$$\int_V c_1(L)^{\dim V} \geq 0$$

holds one obtains the notion of *numerically effective* line bundle (*nef*[2], for short). Kleiman has proved in [39] that a line bundle L over X is nef if and only if for any irreducible curve C on X the following inequality

$$\int_C c_1(L) \geq 0$$

holds ([39, Theorem III.2.1]). This last condition is the definition of nef line bundle used nowadays.

These results justify the generic adjective "positive" associated with the ampleness and nefness conditions for line bundles.

Positivity conditions and curve semistable vector bundles

Hartshorne has introduced in [29] a notion of ampleness for vector bundles of higher rank.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and let X be a projective scheme over $\mathbb{K}$. A vector bundle E over X is *ample* if the line bundle $\mathcal{O}_{\mathrm{Gr}_1(E)}(1)$ over $\mathrm{Gr}_1(E)$ (the projective bundle of rank 1 locally free quotients of E, see [25] or [45]) is ample. Similarly, a vector bundle E over X is *nef* if the line bundle $\mathcal{O}_{\mathrm{Gr}_1(E)}(1)$ over $\mathrm{Gr}_1(E)$ is nef ([45, Definition 6.1.1]). Finally, a vector bundle E over X is *numerically flat* if E and $E^\vee$ are both nef.

On the other hand, in order to construct the *moduli space of vector bundles over a smooth irreducible projective curve*, Mumford has introduced in [53, Definition at page 529] the

[1] In some sense, this colloquial phrase can be formalized, see [45, Theorem 1.4.23].

[2] Other possibilities are, in chronological order, *arithmetically effective*, *numerically eventually free* and *pseudoample* (See [45, Remark 1.4.2]).

so-called (*slope semi*)*stability condition*. Later, Takemoto has extended in [64] this notion on any smooth irreducible projective variety (Definition 1.2.1).

Positivity conditions and semistability for vector bundles over smooth irreducible projective varieties are not "disjoint" properties, in the sense that the first ones determine the second one and *vice versa*. We start reviewing the simplest case: the semistable vector bundles over smooth irreducible projective curves.

Let C be a smooth irreducible projective curve and let E be a rank $r \geq 2$ vector bundle over C. [52, Theorems 3.4 and 3.4'] prove that E is semistable if and only if $E \otimes \det(E)^{-1/r}$ is numerically flat. These proofs follow by another equivalent condition for the semistability of E: E is semistable if and only if the *normalized hyperplane class* $\lambda_1(E) \in N^1(\mathrm{Gr}_1(E))$ is nef (cfr. Equation (4.1) and [52, Theorem 3.1]). Bruzzo and Hernández Ruipérez have generalised in [15] *Miyaoka's criterion* introducing other numerical classes $\lambda_s(E) \in N^1(\mathrm{Gr}_1(Q_{s,E}))$ and $\theta_s(E) \in N^1(\mathrm{Gr}_1(E))$ where $s \in \{1, \ldots, r-1\}$ (cfr. Equations (4.1) and (4.2), respectively). They have proved that the semistability of E implies the nefness of all these numerical classes; and, conversely, if one of these numerical classes is nef then E is semistable ([15, Theorem 1.1]).

Moreover, they have given a generalisation of this criterion to semistable vector bundles over smooth complex projective varieties. In this thesis we extend this criterion to smooth projective varieties defined over an algebraically closed field of characteristic 0 (Theorem 4.2.1). Taking also in account Nakayama's result proved in [55], let X be a smooth complex projective variety and let E be a vector bundle over X. The following statements are equivalent ([55, Theorem 2] and [15, Theorem 1.4]):

a) $\theta_1(E)$ is nef;

b) E is *curve semistable* (see Definition 4.1.1);

c) E is semistable with respect to some polarization H and $c_2(\mathrm{End}(E)) = 0 \in \mathrm{H}^4(X, \mathbb{Q})$;

d) E is semistable with respect to some polarization H and $\displaystyle\int_X c_2(\mathrm{End}(E)) \cdot H^{n-2} = 0$.

The interest in this last result is that curve semistable vector bundles over X are semistable with respect to all polarizations of X. Furthermore, this suggests a generalisation to the *Higgs bundles* setting.

The previous statements also hold on smooth projective varieties X defined over an algebraically closed field of characteristic 0 considering the Chern classes $c_k(E)$ of a vector bundle E as elements of the *Chow groups* $A^k(X)$.

Positivity conditions for Higgs bundles

The Higgs bundles were defined by Hitchin on compact Riemann surfaces in the rank 2 case [33]. It has been Nitsure to extend in [56] this definition to smooth projective curves defined over an algebraically closed field and arbitrary rank. Finally, in [61], Simpson has extended this definition to complex manifolds, and this works also on any smooth scheme. Moreover, all these authors, in the papers cited above, use a *Mumford-Takemoto (semi)stability condition type* for Higgs bundles (see Definition 1.1.1). Finally Bruzzo, Lanza and Lo Giudice have introduced the definition of *curve (semi)stable (Higgs) vector bundle* in [16] (Definition 4.1.1) using the previous conditions of (semi)stability.

Bruzzo and Hernández Ruipérez in [15] investigated how the semistability of a rank r Higgs bundle $\mathfrak{E} = (E, \varphi)$ over X, a smooth complex projective variety, can be encoded by the nefness of numerical classes which are sensitive to the Higgs field. Here we have investigated the same assuming that X is defined over an algebraically closed field of characteristic 0, see Theorems 4.1.2 and 4.1.3, for example.

In order to do this, they have introduced closed subschemes $\mathfrak{Gr}_s(\mathfrak{E})$ of $\mathrm{Gr}_s(E)$ (the *s-th Grassmann bundle of E*), called the *s-th Higgs-Grassmann schemes of* $\mathfrak{E}$, which parametrizes the rank s Higgs quotient bundles of $\mathfrak{E}$. These schemes enjoy a universal property similar to that of the Grassmann bundles.

Let $\mathfrak{Q}_{s,\mathfrak{E}}$ be the restriction of the universal rank s quotient bundle of E to $\mathfrak{Gr}_s(\mathfrak{E})$; this is a rank s Higgs quotient bundles of the pullback of $\mathfrak{E}$ over $\mathfrak{Gr}_s(\mathfrak{E})$. Bruzzo and Hernández Ruipérez have defined the numerical classes $\lambda_s(\mathfrak{E}) \in N^1\left(\mathrm{Gr}_1\left(\mathfrak{Q}_{s,\mathfrak{E}}\right)\right)$ and $\theta_s(\mathfrak{E}) \in N^1\left(\mathfrak{Gr}_1(\mathfrak{E})\right)$ where $s \in \{1, \dots, r-1\}$ (see Equations (4.1) and (4.2), respectively). So $\mathfrak{E}$ is curve semistable if and only if all classes $\lambda_s(\mathfrak{E})$ or $\theta_s(\mathfrak{E})$, equivalently, are nef ([13, Theorem A.5 and Lemma A.6]).

To go onto generalise these properties from the ordinary setting to the Higgs bundles setting, one needs a *numerically flatness conditions for Higgs bundles* analogous to that for ordinary vector bundles. Indeed, referring to the previous statement (b), it implies (c) and this is proved using the notion of numerical flat vector bundles.

This notion, together with other, has been introduced by Bruzzo, Hernández Ruipérez and Graña Otero in [15, 11]. The idea is the following: the first Chern class of E has to satisfy the positivity conditions given by Nakai, Moišezon and Kleiman criteria, *i.e.* $\det(E)$ has to be either ample or nef, respectively. Of course this is not enough if $r \geq 2$, so they require also the *H-ampleness*$\backslash$*H-nefness*[3] of all *universal Higgs quotient bundles* $\mathfrak{Q}_{s,\mathfrak{E}}$

[3] H-ample means *Higgs ample* and H-nef means *Higgs nef*.

recursively (see Definitions 3.2.1 and 3.3.1, respectively). For example, let $r = 3$, then $\mathfrak{E}$ is H-ample\H-nef if and only if by definition the following line bundles are ample\nef in the usual sense: $\det(E)$, $\mathfrak{Q}_{1,\mathfrak{E}}$, $\det(\mathfrak{Q}_{2,\mathfrak{E}})$, $\mathfrak{Q}_{1,\mathfrak{Q}_{2,\mathfrak{E}}}$. Finally, $\mathfrak{E}$ is *H-nflat*[4] if and only if $\mathfrak{E}$ and $\mathfrak{E}^\vee$ are both H-nef.

In this way, where $\varphi = 0$, one has the usual definition of Grassmann bundles, universal quotient bundles and ample\nef\nflat vector bundle.

The curve semistable, H-nef and H-nflat Higgs bundles, over smooth complex projective variety, have been studied by Biswas, Bruzzo, myself, Graña Otero, Gurjar, Hernández Ruipérez, Lanza, Lo Giudice and Peragine in [11, 12, 13, 17, 44, 3, 16, 18, 9, 14]. There, the authors have proved that H-nef Higgs bundles satisfy almost all usual properties of nef vector bundles (see Lemma 3.3.3). In this thesis, we prove that most of these results hold also where the underlying field is algebraically closed of characteristic 0 (see Remark 3.3.4 and Proposition 3.3.7).

About the H-ample Higgs bundles, these have been studied in [10], where we prove their basic properties (see Propositions 3.2.5, 3.2.15, Theorem 3.2.10 and Corollaries 3.2.8 and 3.2.13). We give an application of H-ampleness and H-nefness criteria, as expressed by Theorems 3.2.6 and 3.2.12, to minimal smooth surfaces of general type defined over an algebraically closed field of characteristic 0.

Numerically flatness and curve semistable Higgs bundles

In the complex case it has pointed out by Biswas, Bruzzo and Gurjar in [3] the equivalence between the following facts:

A) let $\mathfrak{E} = (E, \varphi)$ be a curve semistable Higgs bundle over X. Then $\mathfrak{E}$ is semistable with respect to some polarization H and $c_2(\mathrm{End}(E)) = 0 \in \mathrm{H}^4(X, \mathbb{Q})$.

B) the Chern classes of any H-nflat Higgs bundle over X vanish.

Here $\mathrm{End}(E) = E \otimes E^\vee$ is the *adjoint bundle of* E.

It is known that curve semistable Higgs bundles $\mathfrak{E} = (E, \varphi)$ are semistable but it is unknown if this condition implies $c_2(\mathrm{End}(E)) = 0$. This last implication holds for ordinary vector bundles. The best of our knowledge, curve semistable Higgs bundles $\mathfrak{E} = (E, \varphi)$ have $c_2(\mathrm{End}(E))$ vanishes at least in the cases listed in Remark 4.2.6.

[4] H-nflat means *Higgs numerically flat.*

In other words, the equivalence of the previous statement (A) and (B) in the Higgs bundles setting holds (Theorem 4.2.2), but it is unknown whether one of these statements hold in general (cfr. Conjectures 2 and 3). The vanishing of the second Chern class of H-nflat Higgs bundles is the obstruction to prove that all curve semistable Higgs bundles are semistable, and the corresponding adjoint Higgs bundles have second Chern classes equal to 0. The inverse implication holds (Theorem 4.2.4), however it has proved by [15, Theorem 1.3] in the complex setting originally.

In order to state all this in our general setting, we have extended the results of Biswas, Bruzzo, myself, Graña Otero and Gurjar proved in [13, 3] and [9] (see Lemmata 3.3.8, 4.3.4 and Theorem 4.3.5).

Results

The first innovation of this thesis is a new proof of semistability of *tensor product of semistable Higgs sheaves* on smooth projective polarized varieties (X, H) defined over an algebraically closed field $\mathbb{K}$ of characteristic 0 (Theorem 2.2.5). This is a "Lefschetz principle"-type theorem, because it follows from the fact that a Higgs bundle $\mathfrak{E}$ is semistable over X if and only if for an extension field $\mathbb{F}$ of $\mathbb{K}$, the pullback of $\mathfrak{E}$ over $X \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{F})$ is semistable (Lemma 2.2.4). This generalises some results proved by Langton in [43].

Lemma 2.2.4 is the cornerstone which permits us to extend the relations between positivity conditions and semistability for Higgs bundles over smooth projective varieties proved assuming $\mathbb{K} = \mathbb{C}$ to our general assumption on $\mathbb{K}$.

A nice application of this lemma is a new proof of the equivalence between the semistability of $\mathfrak{E}$ and $\mathrm{End}(\mathfrak{E})$ (Lemma 2.3.1). This mimics the proof known in the complex setting: Higgs sheaves have the *Harder-Narasimhan filtration* (see Definition 1.4.1 and Theorem 1.4.2) by [63, Section 3] and the tensor product of semistable Higgs sheaves is semistable by [5, Proposition 4.5] or [34, Theorem 5.4].

These results, together with the fact that semistable Higgs bundles over X with first and second Chern classes vanish have all Chern classes vanish ([42, Corollary 6]), are fundamental in order to prove Theorems 4.2.2 and 4.2.4.

As regards H-ample and H-nef Higgs bundles over X, we begin to prove that the category of H-ample Higgs bundles is closed under finite pullbacks and locally free Higgs quotients (Proposition 3.2.5).

Inspired by [3, Lemma 3.3], we characterize H-ample Higgs bundles via their pullback to smooth irreducible projective curves and the corresponding HN-filtrations (Theorem 3.2.6). As a corollary of this theorem we prove a *Barton-Kleiman-type criterion for H-ampleness* (Corollary 3.2.8). This allows us to prove that the category of H-ample Higgs bundles is closed under extensions and tensor products (Theorems 3.2.10 and 3.2.12, respectively).

Following the same ideas, starting from Corollary 3.2.8, we complete the results contained in [3, Section 3] proving a *Barton-Kleiman-type criterion for H-nefness* (Corollary 3.3.5) and that the category of H-nef Higgs bundles is closed under extensions (Proposition 3.3.7).

We characterize H-nflat Higgs bundles via *Jordan-Hölder filtrations for Higgs sheaves* (Theorem 4.3.5). This was proved by Bruzzo and myself in [9] in the complex setting.

The Jordan-Hölder filtrations of H-nflat Higgs allow us to prove the vanishing of Chern classes in the case where the quotients of the filtration have rank at most 2 (Corollaries 4.3.6 and 4.3.7). The second corollary holds, as we prove the existence of an irreducible component of $\mathfrak{Gr}_1(\mathfrak{E})$ which is a divisor of $\mathrm{Gr}_1(E)$ and surjects onto X, where $\mathfrak{E} = (E, \varphi)$ is a rank 2 Higgs bundle (Proposition 3.1.4; in the complex setting, this is [14, Corollary 4.3]).

Finally, we apply these results to minimal smooth projective varieties X. We consider the so-called *Simpson system* $\mathfrak{S}$ over X, a Higgs bundle studied in [61]. In Section 5.1, we assume that X is a surface and the *Bogomolov-Miyaoka-Yau inequality* (5.1) is saturated by X. Under this assumption we prove that $\mathfrak{S}$ is curve semistable and H-ample (Proposition 5.1.1 and Lemma 5.1.7, respectively) together with other interesting results (Theorem 5.1.3 and Corollary 5.1.4). These extend results proved by Miyaoka in [51] for complex minimal smooth surfaces of general type.

In Section 5.2, we assume $\mathbb{K} = \mathbb{C}$, $\dim X = n \geq 3$ and K_X is ample. Under these assumptions, we prove the stability of $\mathfrak{S}$ with respect to K_X (Theorem 5.2.1) and we give a new proof of the *Guggenheimer-Yau inequality* (5.2) (Theorem 5.2.6). Again, on X which saturates (5.2), we prove other interesting results (Theorem 5.2.7 and Corollary 5.2.8), the H-ampleness and the curve semistability of $\mathfrak{S}$ (Corollary 5.2.2 and Lemma 5.2.3, respectively).

The stability of $\mathfrak{S}$ allow us to give a new proof of the fact that minimal smooth complex projective varieties of general type X which saturate (5.2) and have ample canonical bundle, are uniformized by the complex ball $\mathbb{B}^n$ and their cotangent bundle is ample (Theorem 5.2.9 and Corollary 5.2.4, respectively).

Contents

In Chapter 1 we recall some notions about Higgs sheaves on smooth projective varieties X, defined over an algebraically closed field of characteristic 0. We extend their basic properties, as proved in [34], to our setting. We study the *HN-filtration of Higgs sheaves*, which was introduced by Simpson in [63] originally only in the complex setting; furthermore we prove the so-called *Maximal Property of the HN-Polygon* for Higgs sheaves (see the subsection 1.4.2). This property has been proved by Shatz in [60] for torsion-free coherent sheaves originally.

However, the proofs of some results of this chapter require the semistability of the tensor product of semistable Higgs sheaves. This is available in the complex setting ([5, 34]), therefore in Chapter 2 we extend this result to our setting (Theorem 2.2.5).

In Chapter 3 we recall the notion of Higgs-Grassmann schemes $\mathfrak{Gr}_s(\mathfrak{E}) \subseteq \mathrm{Gr}_s(E)$ of a rank r Higgs bundle $\mathfrak{E} = (\mathfrak{E}, \varphi)$ over X, where $s \in \{1, \ldots, r-1\}$. As wrote above, these schemes are used to provide notions of *H-ample* and *H-nef* Higgs bundles.

We prove basic properties of H-ample Higgs bundles, and criteria for ampleness and numerical effectiveness of Higgs bundles of the Barton-Kleiman type (Corollaries 3.2.8 and 3.3.5, respectively). Both criteria use explicitly the HN-filtration for Higgs bundles over smooth irreducible projective curves. We prove that the notion of H-ampleness is well-behaved with respect to tensor products and extensions (Theorems 3.2.10 and 3.2.12, respectively).

These positivity conditions for Higgs bundles are applied in chapter 4 to study the semistability of particular Higgs bundles over X: the curve semistable Higgs bundles (Definition 4.1.1). We state and prove the equivalence of the curve semistability for vector bundles with other conditions (Theorem 4.2.1). This is possible in the setting of vector bundles over X because in [42] Langer has extended [62, Theorem 2] from $\mathbb{C}$ to $\mathbb{K}$.

In [9] we studied the Chern classes of H-nflat Higgs bundles over complex simply-connected Calabi-Yau varieties. This is made characterizing the H-nflat Higgs bundles via JH-filtrations. In Section 4.3 we generalise this to our setting.

Finally, in Chapter 5 we study the minimal smooth complex projective varieties as described above.

In Appendix A, we recall the notion of *1-numerically flat* Higgs bundle (*1-H-nflat*, for short) and the main properties of these objects. This is made because these objects are cited in Remark 4.2.6.

Notation

All rings are commutative with unit. Analogously, any algebra over a ring is commutative and associative.

By a projective variety X we mean a projective integral scheme of dimension $n \geq 1$ and of finite type, defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, unless otherwise indicated; in particular these schemes are Noetherian, irreducible, reduced and separated over $\mathrm{Spec}(\mathbb{K})$. If $\dim X \in \{1, 2\}$ we shall write projective curve or projective surface, respectively.

We shall denote by E a rank $r \geq 1$ vector bundle over X, *i.e.* a locally free sheaf of $\mathcal{O}_X$-modules on X, while we shall use the script character $\mathcal{E}$ to indicate any coherent sheaf. Somewhere we shall confuse interchangeably vector bundles and locally free sheaves.

As usual the degree d of a vector bundle of rank r (with respect to a fixed polarization H of X) is defined as the degree of its determinant bundle $\bigwedge^{r} E \cong \det(E) \in \mathrm{Pic}(X)$ (with respect to H), while the degree of any coherent sheaf $\mathcal{E}$ is defined using free resolutions; in other words, $\det(\mathcal{E}) \overset{def.}{=} \bigwedge^{\mathrm{rank}(\mathcal{E})} \mathcal{E}^{\vee\vee}$.

Let X be a scheme of finite type over an algebraically closed field of characteristic 0. We denote by $\mathrm{N}^1(X) = \dfrac{\mathrm{Div}(X)}{\equiv_{num}} \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space of real 1-cocycles on X modulo numerical equivalence. We denote by $\mathrm{A}^k(X) = \dfrac{\mathrm{Z}_{n-k}(X)}{\equiv_{rat}}$ the Abelian group of k-cocycles on X modulo rational equivalence.

Chapter 1

Higgs sheaves: an overview

1.1 Basic notions

Let X be a smooth scheme over $\mathbb{K}$, an algebraically closed field of characteristic 0. Let Ω_X^1 be the cotangent bundle of X and let $\Omega_X^k = \bigwedge^k \Omega_X^1$, where $k \in \{1, \dots, n = \dim X\}$.

Definition 1.1.1. A *Higgs sheaf* $\mathfrak{E}$ is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is an $\mathcal{O}_X$-coherent sheaf equipped with a morphism $\varphi\colon \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$ called *Higgs field* such that the composition

$$\varphi \wedge \varphi\colon \mathcal{E} \xrightarrow{\varphi} \mathcal{E} \otimes \Omega_X^1 \xrightarrow{\varphi \otimes \mathrm{Id}} \mathcal{E} \otimes \Omega_X^1 \otimes \Omega_X^1 \to \mathcal{E} \otimes \Omega_X^2$$

vanishes. This last request is called the *integrability condition*. A Higgs subsheaf of $\mathfrak{E}$ is a *φ-invariant subsheaf* $\mathcal{F}$ of $\mathcal{E}$, that is $\varphi(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega_X^1$. A *Higgs quotient* of $\mathfrak{E}$ is a quotient sheaf of $\mathcal{E}$ such that the corresponding kernel is φ-invariant. A *Higgs bundle* is a Higgs sheaf whose underlying coherent sheaf is locally free.

Remark 1.1.2.

a) The Higgs field φ is a global section of $\mathrm{End}(\mathcal{E}) \otimes \Omega_X^1$.

b) If $n = 1$ and $\mathbb{K} = \mathbb{C}$, we have the original definition of Higgs bundle given by Hitchin in [33], since $\Omega_X^1 = K_X$ (the *canonical bundle of* X). $\diamondsuit$

Here we briefly recall some operations on Higgs sheaves which will be useful in the next (see also [58]).

We have a natural dual morphism $\varphi^\vee\colon \mathcal{E}^\vee \otimes (\Omega_X^1)^\vee \to \mathcal{E}^\vee$; the morphism

$$\mathcal{E}^\vee \cong \mathcal{E}^\vee \otimes \mathcal{O}_X \xrightarrow{Id \otimes \mathrm{Tr}^\vee} \mathcal{E}^\vee \otimes \left(\Omega_X^1\right)^\vee \otimes \Omega_X^1 \xrightarrow{\varphi^\vee \otimes \mathrm{Id}} \mathcal{E}^\vee \otimes \Omega_X^1$$

defines a Higgs field on $\mathcal{E}^\vee$ which is denoted, by abuse of notation, as $\varphi^\vee$. The pair $\mathfrak{E}^\vee = (\mathcal{E}^\vee, \varphi^\vee)$ is the *dual* Higgs sheaf of $\mathfrak{E}$.

On the other hand, if Y is another smooth projective variety and $f\colon Y \to X$ is a morphism, on $f^*\mathcal{E}$ one can define the following Higgs field

$$f^*\mathcal{E} \xrightarrow{f^*\varphi} f^*\mathcal{E} \otimes f^*\Omega^1_X \xrightarrow{Id\otimes f^*} f^*\mathcal{E} \otimes \Omega^1_Y$$

which is denoted, by abuse of notation, as $f^*\varphi$. The pair defined by $f^*\mathfrak{E} = (f^*\mathcal{E}, f^*\varphi)$ is the *pullback Higgs sheaf of* $\mathfrak{E}$ *via* f.

Given two Higgs sheaves $\mathfrak{E}_1 = (\mathcal{E}_1, \varphi_1)$ and $\mathfrak{E}_2 = (\mathcal{E}_2, \varphi_2)$ on X, we can construct the direct sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2 = (\mathcal{E}_1 \oplus \mathcal{E}_2, \varphi_1 \oplus \varphi_2 \equiv \mathrm{pr}_1^* \varphi_1 + \mathrm{pr}_2^* \varphi_2)$, where $\mathrm{pr}_k\colon \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}_k$ is the k-th canonical projection; and the tensor product $\mathfrak{E}_1 \otimes \mathfrak{E}_2 = (\mathcal{E}_1 \otimes \mathcal{E}_2, \varphi_1 \otimes \varphi_2 \equiv \varphi_1 \otimes \mathrm{Id}_{\mathcal{E}_2} + Id_{\mathcal{E}_1} \otimes \varphi_2)$, where $Id_{\mathcal{E}_k}$ is the identity automorphism of $\mathcal{E}_k$, in both cases $k \in \{1, 2\}$.

Here we recall the definition of morphisms of Higgs sheaves.

Definition 1.1.3. Let $\mathfrak{E}_1 = (\mathcal{E}_1, \varphi_1)$ and $\mathfrak{E}_2 = (\mathcal{E}_2, \varphi_2)$ be Higgs sheaves. A morphism $f\colon \mathfrak{E}_1 \to \mathfrak{E}_2$ of Higgs sheaves is a morphism of $\mathcal{O}_X$-modules (indicated again as) $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that the following diagram

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "f"] \arrow[d, "\varphi_1"'] & \mathcal{E}_2 \arrow[d, "\varphi_2"] \\
\mathcal{E}_1 \otimes \Omega^1_X \arrow[r, "f\otimes\mathrm{Id}"'] & \mathcal{E}_2 \otimes \Omega^1_X
\end{tikzcd}$$

commutes.

The kernel and the image of morphisms of Higgs sheaves are Higgs sheaves. In fact, if $f\colon \mathfrak{E}_1 \to \mathfrak{E}_2$ is a morphism of Higgs sheaves, let $\mathcal{K} = \ker(f)$ and let $i\colon \mathcal{K} \to \mathfrak{E}_1$ be the obvious inclusion, we have the following commutative diagram (with left exact rows)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r, "i"] \arrow[d, "\varphi_{1|\mathcal{K}}"'] & \mathcal{E}_1 \arrow[r, "f"] \arrow[d, "\varphi_1"'] & \mathcal{E}_2 \arrow[d, "\varphi_2"'] \\
0 \arrow[r] & \mathcal{K} \otimes \Omega^1_X \arrow[r, "i\otimes\mathrm{Id}"'] & \mathcal{E}_1 \otimes \Omega^1_X \arrow[r, "f\otimes\mathrm{Id}"'] & \mathcal{E}_2 \otimes \Omega^1_X
\end{tikzcd}$$

In this way the pair $\mathfrak{K} = (\mathcal{K}, \varphi_{1|\mathcal{K}})$ becomes a Higgs subsheaf of $\mathfrak{E}_1$. Similarly, let $\mathcal{F} = \mathrm{Im}(f)$ and let $j\colon \mathcal{F} \to \mathfrak{E}_2$ be the inclusion morphism, we write $f = j \circ p$ and obtain the following commutative diagram

$$\begin{tikzcd}
\mathfrak{E}_1 \arrow[r, "p"] \arrow[d, "\varphi_1"'] & \mathcal{F} \arrow[r, "j"] \arrow[d, "\varphi_{2|\mathcal{F}}"'] & \mathfrak{E}_2 \arrow[d, "\varphi_2"'] \\
\mathfrak{E}_1 \otimes \Omega^1_X \arrow[r, "p\otimes \mathrm{Id}"'] & \mathcal{F} \otimes \Omega^1_X \arrow[r, "j\otimes \mathrm{Id}"'] & \mathfrak{E}_2 \otimes \Omega^1_X
\end{tikzcd} .$$

$\mathfrak{F} = (\mathcal{F}, \varphi_{2|\mathcal{F}})$ is a Higgs sheaf. Furthermore, from the above diagram it follows that $\mathfrak{F}$ is a Higgs subsheaf of $\mathfrak{E}_2$ and at the same time a Higgs quotient of $\mathfrak{E}_1$.

A short exact sequence of Higgs sheaves (also called an *extension of Higgs sheaves* or a *Higgs extension*)

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{Q} \longrightarrow 0 \tag{1.1}$$

is defined in the obvious way.

1.2 Mumford-Takemoto (semi)stability condition

This section is mainly based on [34].

Let (X, H) be a smooth polarized variety and let $\mathfrak{E} = (\mathcal{E}, \varphi)$ be a torsion-free Higgs sheaf on X, if not otherwise indicated. One defines the degree $\deg(\mathfrak{E})$ and the rank $\mathrm{rank}(\mathfrak{E})$ of $\mathfrak{E}$ simply those of $\mathcal{E}$, respectively. In particular

$$\deg(\mathfrak{E}) = \int_X c_1(\det(\mathcal{E})) \cdot H^{n-1}.$$

One defines the *slope* of $\mathfrak{E}$ as $\mu(\mathfrak{E}) = \dfrac{\deg(\mathfrak{E})}{\mathrm{rank}(\mathfrak{E})} \in \mathbb{Q}$. In a similar way as for sheaves (see [31]) there is a notion of stability for Higgs sheaves (see [5, 34], for example).

Definition 1.2.1. $\mathfrak{E}$ is *semistable* (respectively, *stable*) if $\mu(\mathfrak{F}) \leq \mu(\mathfrak{E})$ (respectively, $\mu(\mathfrak{F}) < \mu(\mathfrak{E})$) for every Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ with $0 < \mathrm{rank}(\mathfrak{F}) < \mathrm{rank}(\mathfrak{E})$. In the other eventuality, $\mathfrak{E}$ is *unstable.*

Example 1.2.2. By convection, any torsion-free Higgs sheaf on X of rank 1 is stable. $\triangle$

Since the notion of (semi)stability for Higgs sheaves makes only reference to Higgs subsheaves, a sheaf could be (semi)stable as a Higgs sheaf, but not as ordinary sheaf, as the next example proves.

Example 1.2.3 ([17, Example 2.9]). Let $\mathfrak{E} = \left(E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \varphi\right)$, where K is the canonical bundle of a smooth projective curve X of genus $g \geq 2$, $K^{\frac{1}{2}}$ is a line bundle over X whose square is K and

$$\varphi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}, \, 1 \in \mathrm{Hom}\left(K^{\frac{1}{2}}, K^{-\frac{1}{2}} \otimes K\right), \, \omega \in \mathrm{H}^0(X, K^2).$$

$\mathfrak{E}$ is a stable Higgs bundle, because there are no subbundles of positive degree preserved by φ. Indeed, let L be a line subbundle of E then L is either $K^{-\frac{1}{2}}$ or $K^{\frac{1}{2}}$. Since

$$\deg\left(K^{\frac{1}{2}}\right) = g - 1 > 0, \, \deg\left(K^{-\frac{1}{2}}\right) = 1 - g < 0$$

L destabilizes E if and only if $L = K^{\frac{1}{2}}$. However $K^{\frac{1}{2}}$ does not destabilize $\mathfrak{E}$, because this is not φ-invariant. △

Remark 1.2.4.

a) Since the previous notions depend on the fixed polarization H of X, we should say H-degree of, H-slope of, H-(semi)stable, H-polystable Higgs sheaf, respectively.

   For simplicity, we shall skip any reference to the fixed polarization H of X if there is no confusion.

b) There exists another possible definition of (semi)stability for Higgs sheaves inspired by Gieseker and Maruyama (semi)stability for vector bundles (cfr. [35]) which will not be used in this thesis. ◊

Any exact sequence of Higgs sheaves as in (1.1) satisfies

$$\mathrm{rank}(\mathfrak{K})(\mu(\mathfrak{E}) - \mu(\mathfrak{K})) + \mathrm{rank}(\mathfrak{Q})(\mu(\mathfrak{E}) - \mu(\mathfrak{Q})) = 0 \tag{1.2}$$

if $\mathrm{rank}(\mathfrak{K}), \mathrm{rank}(\mathfrak{Q}) > 0$ (see [41, Lemma V.7.3]). From equality (1.2) follows that the condition of (semi)stability can be written in terms of Higgs quotient sheaves instead of Higgs subsheaves. As a direct consequence of equality (1.2), we have the following result.

Proposition 1.2.5. *$\mathfrak{E}$ is (semi)stable if $\mu(\mathfrak{E}) \underset{(\leq)}{<} \mu(\mathfrak{Q})$ for every Higgs quotient $\mathfrak{Q}$ of $\mathfrak{E}$ with $0 < \mathrm{rank}(\mathfrak{Q}) < \mathrm{rank}(\mathfrak{E})$.*

Actually, we do not have to consider all Higgs quotient sheaves of a torsion-free Higgs sheaf on X in order to check its (semi)stability.

Proposition 1.2.6.

a) $\mathfrak{E}$ *is (semi)stable if and only if* $\mu(\mathfrak{F}) \underset{(\leq)}{<} \mu(\mathfrak{E})$ *for every Higgs subsheaf* $\mathfrak{F}$ *of* $\mathfrak{E}$ *with* $0 < \operatorname{rank}(\mathfrak{F}) < \operatorname{rank}(\mathfrak{E})$ *such that* $\mathfrak{E}/\mathfrak{F}$ *is a torsion-free Higgs quotient.*

b) $\mathfrak{E}$ *is (semi)stable if and only if* $\mu(\mathfrak{E}) \underset{(\leq)}{<} \mu(\mathfrak{Q})$ *for every torsion-free Higgs quotient* $\mathfrak{Q}$ *with* $0 < \operatorname{rank}(\mathfrak{Q}) < \operatorname{rank}(\mathfrak{E})$.

Remark 1.2.7. A (Higgs) subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{F}$ is a torsion-free (Higgs) quotient is called a *saturated (Higgs) subsheaf of* $\mathcal{E}$. $\diamondsuit$

Proof. Consider an exact sequence of Higgs sheaves as in (1.1) and denote by ψ the Higgs field of $\mathfrak{Q}$. Let $\mathcal{T}$ be the torsion subsheaf of $\mathcal{Q}$, the underlying sheaf to $\mathfrak{Q}$, since[1] $\psi(\mathcal{T}) \subseteq \mathcal{T} \otimes \Omega^1_X$, the pair $\mathfrak{T} = \left(\mathcal{T}, \psi_{|\mathcal{T}}\right)$ is a Higgs subsheaf of $\mathfrak{Q}$[2], with torsion-free Higgs quotient $\overline{\mathfrak{Q}}$. Furthermore $\deg \mathcal{T} \geq 0$. If we define $\overline{\mathfrak{K}}$ as the kernel of the Higgs morphism $\mathfrak{E} \twoheadrightarrow \overline{\mathfrak{Q}}$, we have the following commutative diagram of Higgs sheaves

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathfrak{T} \arrow[d] & \\
0 \arrow[r] & \mathfrak{K} \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d, "Id"'] & \mathfrak{Q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathfrak{K}} \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] & \overline{\mathfrak{Q}} \arrow[r] \arrow[d] & 0 \\
 & \overline{\mathfrak{K}}/\mathfrak{K} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

where all rows and columns are exact. $\mathfrak{K}$ is a Higgs subsheaf of $\overline{\mathfrak{K}}$ by the couniversal property of kernels of morphisms, and $\overline{\mathfrak{K}}/\mathfrak{K} \cong \mathfrak{T}$ by the Snake Lemma. From all this we

[1] ψ is a morphism of $\mathcal{O}_X$-modules, then $\psi(\mathcal{T})$ is contained in the torsion part of $\mathcal{F} \otimes \Omega^1_X$, which is exactly $\mathcal{T} \otimes \Omega^1_X$ because Ω^1_X is locally free.

[2] In general, the torsion subsheaf of a Higgs sheaf is always a Higgs subsheaf. Indeed, let $\mathcal{T}$ be a torsion subsheaf of $\mathfrak{E}$, then $\varphi(\mathcal{T})$ is contained in the torsion part of $\mathfrak{E} \otimes \Omega^1_X$ which is exactly $\mathcal{T} \otimes \Omega^1_X$, because Ω^1_X is locally free.

have

$$\deg(\mathfrak{Q}) = \deg(\mathfrak{T}) + \deg\left(\overline{\mathfrak{Q}}\right) \geq \deg\left(\overline{\mathfrak{Q}}\right),$$
$$\deg\left(\overline{\mathfrak{K}}\right) = \deg(\mathfrak{T}) + \deg(\mathfrak{K}) \geq \deg(\mathfrak{K}).$$

Since $\mathfrak{T}$ is torsion, $\operatorname{rank}(\mathcal{Q}) = \operatorname{rank}\left(\overline{\mathcal{Q}}\right)$ and $\operatorname{rank}(\mathfrak{K}) = \operatorname{rank}\left(\overline{\mathfrak{K}}\right)$. So that

$$\mu(\mathfrak{E}) \underset{(\geq)}{<} \mu\left(\overline{\mathfrak{K}}\right) = \mu(\mathcal{K}),$$
$$\mu(\mathfrak{E}) \underset{(\leq)}{<} \mu\left(\overline{\mathfrak{Q}}\right) = \mu(\mathfrak{Q}),$$

and $\mathfrak{E}$ is (semi)stable by Definition 1.2.1 or by Proposition 1.2.5. Q.e.d.

Proposition 1.2.8. *Let $\mathfrak{L}$ be a Higgs line bundle over X. Then $\mathfrak{E} \otimes \mathfrak{L}$ is (semi)stable if and only if $\mathfrak{E}$ is (semi)stable.*

Proof. Note that $\mu(\mathfrak{E} \otimes \mathfrak{L}) = \mu(\mathfrak{E}) + \deg(\mathfrak{L})$. Let $\mathfrak{E}$ be (semi)stable and let $\mathfrak{F}$ be a Higgs subsheaf of $\mathfrak{E} \otimes \mathfrak{L}$ with $0 < \operatorname{rank}(\mathfrak{F}) < \operatorname{rank}(\mathfrak{E} \otimes \mathfrak{L})$, then

$$\mu(\mathfrak{F}) = \mu\left(\mathfrak{F} \otimes \mathfrak{L}^\vee\right) + \deg(\mathfrak{L}) \underset{(\leq)}{<} \mu(\mathfrak{E}) + \deg(\mathfrak{L}) = \mu(\mathfrak{E} \otimes \mathfrak{L})$$

and $\mathfrak{E} \otimes \mathfrak{L}$ is (semi)stable. The proof of the inverse implication is obvious. Q.e.d.

Since $\det(\mathcal{E}) \cong \left(\bigwedge^r \mathcal{E}\right)^{\vee\vee}$, then φ defines a Higgs field $\det(\varphi)$ on $\det(\mathcal{E})$ *i.e.* $\det(\mathfrak{E}) = (\det(\mathcal{E}), \det(\varphi))$ is a locally free Higgs sheaf, hence it is a Higgs line bundle. On the other hand, by [31, Corollary 1.2] the determinant bundle of any torsion-free sheaf $\mathcal{F}$ satisfies $\det\left(\mathcal{F}^\vee\right) = \det(\mathcal{F})^\vee$. Consequently, if $\mathcal{F}$ is torsion-free then $\mu(\mathcal{F}) = -\mu\left(\mathcal{F}^\vee\right)$ and we have the following result.

Lemma 1.2.9. *$\mathfrak{E}$ is (semi)stable if and only if $\mathfrak{E}^\vee$ is (semi)stable.*

Proof. Assume first that $\mathfrak{E}^\vee$ is (semi)stable and consider a short exact sequence of Higgs sheaves (1.1) on X with $\mathfrak{Q}$ torsion-free and $0 < \operatorname{rank}(\mathfrak{Q}) < \operatorname{rank}(\mathfrak{E})$. Dualizing it, we obtain the following left exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{Q}^\vee \longrightarrow \mathfrak{E}^\vee \longrightarrow \mathfrak{K}^\vee .$$

Since $\mathfrak{E}$ and $\mathfrak{Q}$ are both torsion-free, we have from the above sequence

$$-\mu(\mathfrak{E}) = \mu\left(\mathfrak{E}^\vee\right) \underset{(\leq)}{<} \mu\left(\mathfrak{Q}^\vee\right) = -\mu(\mathfrak{Q})$$

i.e. $\mathfrak{E}$ is (semi)stable by Proposition 1.2.5.
Now let $\mathfrak{E}$ be (semi)stable and consider a short exact sequence of Higgs sheaves on X

$$0 \longrightarrow \mathfrak{K}_0 \longrightarrow \mathfrak{E}^\vee \longrightarrow \mathfrak{Q}_0 \longrightarrow 0$$

with $\mathfrak{Q}_0$ torsion-free and $0 < \operatorname{rank}(\mathfrak{Q}_0) < \operatorname{rank}(\mathfrak{E}^\vee)$. Dualizing this sequence, we have again a left exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{Q}_0^\vee \longrightarrow \mathfrak{E}^{\vee\vee} \longrightarrow \mathfrak{K}_0^\vee \,.$$

Since $\mathfrak{E}$ is torsion-free, the natural morphism $\delta\colon \mathfrak{E} \to \mathfrak{E}^{\vee\vee}$ is injective and thus we have $\mathfrak{E} \cong \operatorname{Im}(\delta)$ and this is a Higgs subsheaf of $\mathfrak{E}^{\vee\vee}$. Since there may be no confusion, we will write simply $\mathfrak{E}$ instead of $\operatorname{Im}(\delta)$. After defining $\mathfrak{H}_0 = \mathfrak{E} \cap \mathfrak{Q}_0^\vee$ and $\mathfrak{H}_1 = \mathfrak{E}/\mathfrak{H}_0$, we have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathfrak{H}_0 \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d, "\sigma"] & \mathfrak{H}_1 \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{Q}_0^\vee \arrow[r, "p^\vee"] \arrow[d, "\varpi"] & \mathfrak{E}^{\vee\vee} \arrow[r] \arrow[d, "\pi"] & \mathfrak{K}_0^\vee & \\
 & \mathfrak{Q}_0^\vee/\mathfrak{H}_0 \arrow[d] & \mathfrak{E}^{\vee\vee}/\mathfrak{E} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the columns and the first row are exact and *a posteriori* also the second and third row; $p^\vee$ is the obvious inclusion. Since $\varpi(\mathfrak{H}_0) = 0$ and $p^\vee(\pi(\mathfrak{H}_0)) = 0$, by the universal property of quotients there exists a unique morphism $q\colon \mathfrak{Q}_0^\vee/\mathfrak{H}_0 \to \mathfrak{E}^{\vee\vee}/\mathfrak{E}$ which makes commutative the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathfrak{H}_0 \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d, "\sigma"] & \mathfrak{H}_1 \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{Q}_0^\vee \arrow[r, "p^\vee"] \arrow[d, "\varpi"] & \mathfrak{E}^{\vee\vee} \arrow[r] \arrow[d, "\pi"] & \mathfrak{K}_0^\vee & \\
 & \mathfrak{Q}_0^\vee/\mathfrak{H}_0 \arrow[r, "q"] \arrow[d] & \mathfrak{E}^{\vee\vee}/\mathfrak{E} \arrow[r] \arrow[d] & \mathfrak{H}_2 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

q is a monomorphism of Higgs sheaves. Indeed, reasoning on the stalks, let $x \in X$ and let $t \in \mathfrak{Q}_{0,x}^{\vee}/\mathfrak{H}_{0,x}$, there exists $s \in \mathfrak{Q}_{0,x}^{\vee}$ such that $\varpi_x(s) = t$, then

$$0 = q_x(t) = q_x(\varpi_x(s)) = \pi_x\left((p^{\vee})_x(s)\right) = \pi_x(s) \overset{def.}{\iff} s \in \mathfrak{E}_x \Rightarrow s \in \mathfrak{H}_{0,x} \overset{def.}{\iff} t = 0,$$

in other words, q is a monomorphism of Higgs sheaves. It follows that $\mathfrak{H}_2$ is the Higgs quotient of $\mathfrak{E}^{\vee\vee}/\mathfrak{E}$ over $\mathfrak{Q}_0^{\vee}/\mathfrak{H}_0$, and the bottom row is exact also on the left.

By construction, $\mathfrak{E}^{\vee\vee}/\mathfrak{E}$ is a torsion Higgs sheaf; by [57, Corollary at page 75] there exists a codimension 2 closed subset Z of X such that $\mathfrak{E}_{|X\setminus Z}$ is a Higgs bundle; and it follows from this that $X \setminus Z \subseteq \operatorname{Supp}\left(\mathfrak{E}^{\vee\vee}/\mathfrak{E}\right)$. The same reasoning holds for $\mathfrak{Q}_0^{\vee}/\mathfrak{H}_0$, *i.e.* $\operatorname{codim}_X \operatorname{Supp}\left(\mathfrak{Q}_0^{\vee}/\mathfrak{H}_0\right) \geq 2$. Since [41, Proposition VII.6.14] works also in the present hypotheses:

$$\deg\left(\mathfrak{Q}_0^{\vee}/\mathfrak{H}_0\right) = 0 \Rightarrow \deg(\mathfrak{H}_0) = \deg\left(\mathfrak{Q}_0^{\vee}\right),$$

furthermore, since $\mathfrak{Q}_0^{\vee}/\mathfrak{H}_0$ is a torsion Higgs sheaf:

$$\operatorname{rank}\left(\mathfrak{Q}_0^{\vee}/\mathfrak{H}_0\right) = 0 \Rightarrow \operatorname{rank}(\mathfrak{H}_0) = \operatorname{rank}\left(\mathfrak{Q}_0^{\vee}\right).$$

By definition and hypotheses:

$$-\mu(\mathfrak{Q}_0) = \mu\left(\mathfrak{Q}_0^{\vee}\right) = \mu(\mathfrak{H}_0) \underset{(\leq)}{<} \mu\left(\mathfrak{E}\right) = -\mu\left(\mathfrak{E}^{\vee}\right)$$
$$\mu(\mathfrak{Q}_0) \underset{(\geq)}{<} \mu\left(\mathfrak{E}^{\vee}\right),$$

and by Proposition 1.2.5 we have the claim. Q.e.d.

Corollary 1.2.10. *$\mathfrak{E}$ is (semi)stable if and only if the Higgs sheaf $\mathfrak{E}^{\vee\vee}$ is (semi)stable.*

As the definition of (semi)stability for Higgs sheaves uses only proper Higgs subsheaves of positive rank, but it may be reformulated in terms of Higgs subsheaves of arbitrary positive rank and not necessarily proper.

Proposition 1.2.11.

a) $\mathfrak{E}$ is semistable if and only if $\mu(\mathfrak{F}) \leq \mu(\mathfrak{E})$ for every Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ with $0 < \operatorname{rank}(\mathfrak{F}) \leq \operatorname{rank}(\mathfrak{E})$.

b) $\mathfrak{E}$ is semistable if and only if $\mu(\mathfrak{E}) \leq \mu(\mathfrak{Q})$ for every Higgs quotient $\mathfrak{Q}$ of $\mathfrak{E}$ with $0 < \operatorname{rank}(\mathfrak{Q}) \leq \operatorname{rank}(\mathfrak{E})$.

1.3 Properties of semistable Higgs sheaves

In a similar way to the case of vector bundles, we have the following results concerning the first and second Chern classes, direct sums and short exact sequences of semistable Higgs sheaves.

We start from the famous and very important *Bogomolov inequality for Higgs sheaves*. We advice that, by [48, Theorem 1.9 and Section 11], we can apply the usual theory of Chern classes for locally free sheaves to coherent sheaves on X. In a sense which be explained in chapter 4, this is the starting point for the topics studied in this thesis.

Theorem 1.3.1 ([42, Theorem 7]). *Let $\mathfrak{E} = (\mathcal{E}, \varphi)$ be a rank r semistable Higgs sheaf on X, then*

$$\int_X \left(c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 \right) \cdot H^{n-2} \geq 0.$$

We refer to the original paper for a proof, if one assumes $\mathbb{K} = \mathbb{C}$ and $\mathcal{E}$ is locally free the previous theorem is [61, Proposition 3.4].

Theorem 1.3.2. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be torsion-free Higgs sheaves on X. Then $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is semistable if and only if $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are both semistable with $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2) = \mu$.*

Proof. Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be semistable and let $\mathfrak{F}$ be a torsion-free Higgs subsheaf of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ with $0 < \operatorname{rank}(\mathfrak{F}) < \operatorname{rank}(\mathfrak{E}_1 \oplus \mathfrak{E}_2)$. Then we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{F}_1 \arrow[r] \arrow[d] & \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{E}_1 \arrow[r] & \mathfrak{E}_1 \oplus \mathfrak{E}_2 \arrow[r, "pr_2"'] & \mathfrak{E}_2 \arrow[r] & 0
\end{tikzcd}$$

where the rows and the columns are exacts, $\mathfrak{F}_1 = \mathfrak{E}_1 \cap \mathfrak{F}$ and $\mathfrak{F}_2$ is the image of $\mathfrak{F}_1$ via pr_2. Trivially we have:

$$\deg(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \deg(\mathfrak{E}_1) + \deg(\mathfrak{E}_2);$$

since $\mathfrak{E}_1$ and $\mathfrak{E}_2$ have the same slope μ, by Formula (1.2) $\mu(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \mu$; by hypothesis:

$$\forall k \in \{1, 2\}, \ \deg(\mathfrak{F}_k) \leq \mu \operatorname{rank}(\mathfrak{F}_k),$$

then

$$\mu(\mathfrak{F}) = \frac{\deg(\mathfrak{F}_1) + \deg(\mathfrak{F}_2)}{\operatorname{rank}(\mathfrak{F}_1) + \operatorname{rank}(\mathfrak{F}_2)} \leq \mu.$$

$\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is semistable. *Vice versa*, let $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ be semistable, since $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are both proper torsion-free Higgs subsheaves and torsion-free Higgs quotient sheaves of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ of positive rank, we have $\mu(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \mu$. Let $\mathfrak{Q}_k$ be a torsion-free Higgs quotient of $\mathfrak{E}_k$, then by Proposition 1.2.6.b:

$$\mu(\mathfrak{Q}_k) \underset{(\geq)}{<} \mu(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \mu(\mathfrak{E}_k)$$

i.e. $\mathfrak{E}_k$ is semistable, where $k \in \{1, 2\}$. Q.e.d.

Definition 1.3.3. $\mathfrak{E}$ is *polystable* if it is a direct sum of stable Higgs sheaves having the same slope.

Before starting the study of extensions of semistable Higgs sheaves, we need the following properties of morphisms of semistable (torsion-free) Higgs sheaves.

Proposition 1.3.4. *Let $f\colon \mathfrak{E}_1 \to \mathfrak{E}_2$ be a morphism of semistable torsion-free Higgs sheaves on X. Then we have the following statements.*

a) If $\mu(\mathfrak{E}_1) > \mu(\mathfrak{E}_2)$, then $f = 0$ (f is the zero morphism).

b) If $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$ and $\mathfrak{E}_1$ is stable, then $\operatorname{rank}(\mathfrak{E}_1) = \operatorname{rank}(\operatorname{Im}(f))$ *and f is injective unless $f = 0$.*

c) If $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$ and $\mathfrak{E}_2$ is stable, then $\operatorname{rank}(\mathfrak{E}_2) = \operatorname{rank}(\operatorname{Im}(f))$ *and f is generically surjective unless $f = 0$.*

d) If $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, and $f \neq 0$ then $\ker(f)$ *and* $\operatorname{Im}(f)$ *are semistable Higgs sheaves of the same common slope.*

e) If $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are stable, then f is an isomorphism unless $f = 0$.

In the following proofs: $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are both semistable with slopes μ_1 and μ_2 and ranks r_1 and r_2, respectively. Let $\mathfrak{K} = \ker(f), \mathfrak{F} = \operatorname{Im}(f)$, then $\mathfrak{F}$ is a torsion-free quotient Higgs sheaf of $\mathfrak{E}_1$ and a Higgs subsheaf of $\mathfrak{E}_2$.

Proof. (a). Assuming that $f \neq 0$ then

$$\mu(\mathfrak{F}) \leq \mu_2 < \mu_1 \leq \mu(\mathfrak{F})$$

which is an absurd. Then f must be the zero morphism.
(b). Assuming that $f \neq 0$ and $r_1 > \operatorname{rank}(\mathfrak{F})$, by hypotheses:

$$\mu(\mathfrak{F}) \leq \mu_2 = \mu_1 < \mu(\mathfrak{F})$$

which is an absurd. Then $\operatorname{rank}(\mathfrak{F})$ must be r_1 and f is injective.

(c). Assuming that $f \neq 0$ and $r_2 > \operatorname{rank}(\mathfrak{F})$, by hypotheses:

$$\mu(\mathfrak{F}) < \mu_2 = \mu_1 \leq \mu(\mathfrak{F})$$

which is an absurd. Then $\operatorname{rank}(\mathfrak{F})$ must be r_2 and there exists an open dense subset U[3] of X such that for any $x \in U$, $f_x \colon \mathfrak{E}_{1,x} \to \mathfrak{E}_{2,x}$ is surjective.

(d). Since $f \neq 0$ then $\mathfrak{F}$ is a Higgs subsheaf of $\mathfrak{E}_2$. By hypothesis $\mu(\mathfrak{F}) \leq \mu$, the common value of μ_1 and μ_2. However $\mathfrak{F} \cong \mathfrak{E}_1/\mathfrak{K}$ and therefore $\mu(\mathfrak{F}) = \mu(\mathfrak{E}_1/\mathfrak{K}) \geq \mu$ by Proposition 1.2.11.b and we have $\mu(\mathfrak{F}) = \mu$. Considering the short exact sequence

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{E}_1 \longrightarrow \mathfrak{E}_1/\mathfrak{K} \longrightarrow 0,$$

by Equation (1.2) we have $\mu(\mathfrak{K}) = \mu$. Finally, since any Higgs subsheaf $\mathfrak{G}$ of $\mathfrak{K}$ is a Higgs subsheaf of $\mathfrak{E}_1$, we have

$$\mu(\mathfrak{G}) \leq \mu = \mu(\mathfrak{K}),$$

i.e. $\mathfrak{K}$ is semistable. Analogously we prove the semistability of $\mathfrak{F}$.

(e). Assuming that $f \neq 0$, by items b and c f is injective, $\operatorname{rank}(\mathfrak{F}) = \operatorname{rank}(\mathfrak{E}_1) = \operatorname{rank}(\mathfrak{E}_2)$ and $\mu(\mathfrak{F}) = \mu(\mathfrak{E}_2)$. The stability hypothesis forces $\mathfrak{F} = \mathfrak{E}_2$, f is surjective so f is an isomorphism. Q.e.d.

Due to Propositions 1.2.11 and 1.3.4 we are in a position to study the extensions of semistable Higgs sheaves.

Lemma 1.3.5. *Let* $0 \longrightarrow \mathfrak{E}_1 \xrightarrow{i} \mathfrak{E} \xrightarrow{p} \mathfrak{E}_2 \longrightarrow 0$ *be a short exact sequence of torsion-free Higgs sheaves on* X*. If* $\mathfrak{E}_1$ *and* $\mathfrak{E}_2$ *are both semistable with* $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2) = \mu$ *then* $\mathfrak{E}$ *is semistable with the same slope* μ*.*

Proof. Using Equation (1.2) we have $\mu(\mathfrak{E}) = \mu$. If $\mathfrak{E}$ is not semistable then there exists by Proposition 1.2.11 a non-zero Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ such that $\mu(\mathfrak{F}) > \mu$. Without loss of generality, we can assume that $\mathfrak{F}$ is semistable[4]. One has a morphism $f \colon \mathfrak{F} \to \mathfrak{E}_2$ induced by p and $\mu(\mathfrak{F}) > \mu(\mathfrak{E}_2)$, by Proposition 1.3.4.a $f = 0$. Noting that $\mathfrak{E}_1 = \ker(p)$,

[3]By [57, Corollary at page 75] there exists a codimension 2 closed subset Z of X such that $\mathfrak{E}_{k|X\setminus Z}$ are Higgs bundles and we set $U = X \setminus Z$, where $k \in \{1, 2\}$.

[4]If it is not, we choose a non-zero Higgs subsheaf $\mathfrak{F}_1$ of $\mathfrak{F}$ with $0 < \operatorname{rank}(\mathfrak{F}_1) < \operatorname{rank}(\mathfrak{F})$ and $\mu(\mathfrak{F}_1) > \mu(\mathfrak{F})$. If $\mathfrak{F}_1$ is semistable we finish, otherwise we repeat this reasoning until we find a semistable non-zero Higgs subsheaf of $\mathfrak{F}$. All this works because in the worst case we find a Higgs line subbundle of $\mathfrak{E}$ which is stable by Example 1.2.2.

by couniversal property of kernels there exists a unique morphism $g\colon \mathfrak{F} \to \mathfrak{E}_1$ such that $i \circ g$ is the inclusion of $\mathfrak{F}$ in $\mathfrak{E}$. Again, by Proposition 1.3.4.a $g = 0$ and therefore $\mathfrak{F}$ is the zero subsheaf of $\mathfrak{E}$: this is a contradiction with the instability of $\mathfrak{E}$. By all this, we have the claim. Q.e.d.

1.4 Harder-Narasimhan filtration for Higgs sheaves

This section is mainly based on [47].

We start by recalling the main definitions of this section.

Definition 1.4.1. Let $\mathfrak{E}$ be a torsion-free Higgs sheaf. A filtration in Higgs subsheaves

$$\{0\} = \mathfrak{E}_0 \subsetneqq \mathfrak{E}_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}_{m-1} \subsetneqq \mathfrak{E}_m = \mathfrak{E}, \tag{1.3}$$

is called an *Harder-Narasimhan filtration of* $\mathfrak{E}$ (*HN-filtration*, for short) if the successive Higgs quotient sheaves $\mathfrak{E}_i/\mathfrak{E}_{i-1}$ are semistable for any $i \in \{1, \ldots, m\}$, and the sequence $\mu_i = \mu(\mathfrak{E}_i/\mathfrak{E}_{i-1})$ is strictly decreasing.

Theorem 1.4.2. *There exists a unique HN-filtration for* $\mathfrak{E}$.

In order to prove the previous theorem, we need the following preliminary lemma.

Lemma 1.4.3. *There exists an integer number* $N(\mathfrak{E})$ *such that for every Higgs subsheaf* $\mathfrak{F}$ *of* $\mathfrak{E}$ *we have* $\deg(\mathfrak{F}) \leq N(\mathfrak{E})$.

Proof. Let $\operatorname{rank}(\mathfrak{E}) = r$. If $r = 1$, by Example 1.2.2 and Proposition 1.2.11.a, for any Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ with $\operatorname{rank}(\mathfrak{F}) = 1$ we have $\deg(\mathfrak{F}) \leq \deg(\mathfrak{E})$; therefore we get $N(\mathfrak{E}) = \max\{0, \deg(\mathfrak{E})\}$. Let $r \geq 2$, then we choose a non-zero Higgs subsheaf $\mathfrak{K}$ of $\mathfrak{E}$ such that the corresponding Higgs quotient $\mathfrak{Q}$ is torsion-free and we obtain the usual short exact sequence (1.1) with $0 < \operatorname{rank}(\mathfrak{K}), \operatorname{rank}(\mathfrak{Q}) < r$. Let $\mathfrak{F}$ be a Higgs subsheaf of $\mathfrak{E}$; we set $\mathfrak{F}_1 = \mathfrak{K} \cap \mathfrak{F}$. Let $\mathfrak{F}_2$ be the image of $\mathfrak{F}_1$ in $\mathfrak{Q}$. Since $\operatorname{rank}(\mathfrak{F}_1) \leq \operatorname{rank}(\mathfrak{K})$ and $\operatorname{rank}(\mathfrak{F}_2) \leq \operatorname{rank}(\mathfrak{Q})$, by induction:

$$\deg(\mathfrak{F}) = \deg(\mathfrak{F}_1) + \deg(\mathfrak{F}_2) \leq N(\mathfrak{K}) + N(\mathfrak{Q}).$$

Then we let $N(\mathfrak{E}) = \min\{N(\mathfrak{K}) + N(\mathfrak{Q})\}$, where $\mathfrak{K}$ and $\mathfrak{Q}$ range in the sets of Higgs subsheaves and the corresponding Higgs quotient (torsion-free) sheaves of $\mathfrak{E}$, respectively. From all this, we have the claim. Q.e.d.

Proof of Theorem 1.4.2. If $\mathfrak{E}$ is semistable then $0 \subsetneqq \mathfrak{E}$ is the HN-filtration of $\mathfrak{E}$, in particular this happens if $\operatorname{rank}(\mathfrak{E}) = r = 1$ by Example 1.2.2. Otherwise, let $r \geq 2$ and let

$$S = \{\mu(\mathfrak{F}) \in \mathbb{Q} \mid \mathfrak{F} \text{ is a torsion-free Higgs subsheaf of } \mathfrak{E} \text{ with } 0 < \operatorname{rank}(\mathfrak{F}) \leq \operatorname{rank}(\mathfrak{E})\}.$$

It follows from the previous lemma that S has a largest element μ. Then there exists a positive rank torsion-free Higgs subsheaf $\mathfrak{E}_1$ of $\mathfrak{E}$ such that $\mu(\mathfrak{E}_1) = \mu$ and its rank is as large as possible. Moreover, $\mathfrak{E}_1$ is semistable: it cannot admit a Higgs subsheaf of larger slope, because that would contradict the maximality of μ. Let $\widetilde{\mathfrak{E}} = \mathfrak{E}/\mathfrak{E}_1$, of course $1 \leq \operatorname{rank}\left(\widetilde{\mathfrak{E}}\right) < r$. By induction it follows that $\widetilde{\mathfrak{E}}$ admits an HN-filtration

$$\{0\} = \widetilde{\mathfrak{E}}_0 \subsetneqq \widetilde{\mathfrak{E}}_1 \subsetneqq \ldots \subsetneqq \widetilde{\mathfrak{E}}_{l-1} \subsetneqq \widetilde{\mathfrak{E}}_l = \widetilde{\mathfrak{E}},$$

with $j \in \{1, \ldots, l\}$, $\widetilde{\mu}_j = \mu\left(\widetilde{\mathfrak{E}}_j/\widetilde{\mathfrak{E}}_{j-1}\right)$. For any $j \in \{0, \ldots, l\}$ let $\mathfrak{E}_{j+1}$ be the inverse image of $\widetilde{\mathfrak{E}}_j$ in $\mathfrak{E}$, in this way we have a filtration

$$\{0\} = \mathfrak{E}_0 \subsetneqq \mathfrak{E}_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}_l \subsetneqq \mathfrak{E}_{l+1} = \mathfrak{E}.$$

By construction:

$$\forall j \in \{1, \ldots, l\},\ \mathfrak{E}_{j+1}/\mathfrak{E}_j \cong (\mathfrak{E}_{j+1}/\mathfrak{E}_1) / (\mathfrak{E}_j/\mathfrak{E}_1) = \widetilde{\mathfrak{E}}_j/\widetilde{\mathfrak{E}}_{j-1}$$

and these Higgs quotient sheaves are semistable of slopes $\widetilde{\mu}_j$, respectively; in particular $\widetilde{\mu}_1 > \ldots > \widetilde{\mu}_l$. If $\widetilde{\mu}_1 \geq \mu$ then we consider the short exact sequence

$$0 \longrightarrow \mathfrak{E}_1 \longrightarrow \mathfrak{E}_2 \longrightarrow \widetilde{\mathfrak{E}}_1 \longrightarrow 0,$$

$$\mu(\mathfrak{E}_2) = \frac{\deg(\mathfrak{E}_1) + \deg\left(\widetilde{\mathfrak{E}}_1\right)}{\operatorname{rank}(\mathfrak{E}_1) + \operatorname{rank}\left(\widetilde{\mathfrak{E}}_1\right)} \leq \frac{\mu \operatorname{rank}(\mathfrak{E}_1) + \widetilde{\mu}_1 \operatorname{rank}\left(\widetilde{\mathfrak{E}}_1\right)}{\operatorname{rank}(\mathfrak{E}_1) + \operatorname{rank}\left(\widetilde{\mathfrak{E}}_1\right)} \leq \widetilde{\mu}_1$$

$$\mu(\mathfrak{E}_2) = \frac{\deg(\mathfrak{E}_1) + \deg\left(\widetilde{\mathfrak{E}}_1\right)}{\operatorname{rank}(\mathfrak{E}_1) + \operatorname{rank}\left(\widetilde{\mathfrak{E}}_1\right)} \geq \frac{\deg(\mathfrak{E}_1) + \deg\left(\widetilde{\mathfrak{E}}_1\right)}{\frac{1}{\mu}\deg(\mathfrak{E}_1) + \frac{1}{\widetilde{\mu}_1}\deg\left(\widetilde{\mathfrak{E}}_1\right)} \geq \mu$$

i.e. $\mu = \mu(\mathfrak{E}_1) \leq \mu(\mathfrak{E}_2) \leq \mu\left(\widetilde{\mathfrak{E}}_1\right) = \widetilde{\mu}_1$, but this is a contradiction with the maximality of μ in S. Therefore $\mu > \widetilde{\mu}_1 > \ldots > \widetilde{\mu}_l$, in other words $\mathfrak{E}$ admits an HN-filtration.

Let $\mathfrak{E}_1'$ be another positive rank torsion-free Higgs subsheaf of $\mathfrak{E}$ such that $\mu(\mathfrak{E}_1') = \mu$; by the same reasoning $\mathfrak{E}_1'$ is semistable. Let $p\colon \mathfrak{E} \to \mathfrak{E}/\mathfrak{E}_1'$, if $\mathfrak{E}_1 \neq \mathfrak{E}_1'$ then $p(\mathfrak{E}_1) \neq 0$, and we consider the short exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{E}_1 \xrightarrow{p} p(\mathfrak{E}_1) \longrightarrow 0;$$

by Proposition 1.2.11 $\mu(\mathfrak{K}) \leq \mu \leq \mu(p(\mathfrak{E}_1))$. On the other hand, we have the following short exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{E}_1' \longrightarrow \mathfrak{F} \xrightarrow{p} p(\mathfrak{E}_1) \longrightarrow 0;$$

where $\mathfrak{F}$ is the inverse image of $p(\mathfrak{F}) = p(\mathfrak{E}_1)$ in $\mathfrak{E}$. By construction $\mu(\mathfrak{F}) < \mu = \mu(\mathfrak{E}_1')$, in other words:

$$\mu(\mathfrak{F}) - \mu(\mathfrak{E}_1) < 0 \iff \mu(\mathfrak{F}) - \mu(p(\mathfrak{E}_1)) > 0$$

by Equation (1.2), *i.e.* $\mu(p(\mathfrak{E}_1)) < \mu(\mathfrak{E}_1')$ and this is a contradiction by Proposition 1.2.11.b.

From all this, by induction $\mathfrak{E}$ admits a unique HN-filtration. Q.e.d.

Definition 1.4.4. The torsion-free Higgs sheaf $\mathfrak{E}_1$ is called the *maximal destabilizing Higgs subsheaf of* $\mathfrak{E}$.

This torsion-free Higgs subsheaf can be characterized in the following alternative way.

Lemma 1.4.5 (cfr. [4, Lemma 4.2])**.** *Let*

$$\{0\} = \mathfrak{E}_0 \subsetneq \mathfrak{E}_1 \subsetneq \ldots \subsetneq \mathfrak{E}_{m-1} \subsetneq \mathfrak{E}_m = \mathfrak{E}$$

be a filtration of torsion-free Higgs subsheaves such that

- $\mathfrak{E}_1$ *is semistable;*
- *for any* $i \in \{1, \ldots, m\}, \mathfrak{E}_i/\mathfrak{E}_{i-1}$ *is semistable and* $\mu(\mathfrak{E}_1) > \mu(\mathfrak{E}_i/\mathfrak{E}_{i-1})$.

Then $\mathfrak{E}_1$ *is the maximal destabilizing Higgs subsheaf of* $\mathfrak{E}$.

Proof. Let $\mathfrak{E}_1'$ be a torsion-free semistable Higgs subsheaf of $\mathfrak{E}$ such that $\mu(\mathfrak{E}_1') \geq \mu(\mathfrak{E}_1)$. Considering the morphism

$$\mathfrak{E}_1' \hookrightarrow \mathfrak{E}_m = \mathfrak{E} \twoheadrightarrow \mathfrak{E}_m/\mathfrak{E}_{m-1},$$

since $\mu(\mathfrak{E}_1') > \mu(\mathfrak{E}_1) \geq \mu(\mathfrak{E}_m/\mathfrak{E}_{m-1})$, by Proposition 1.3.4.a this morphism is the zero morphism. In other words, $\mathfrak{E}_1' \subseteq \mathfrak{E}_{m-1}$. Repeating this construction $m-1$ times, we have $\mathfrak{E}_1' \subseteq \mathfrak{E}_1$ and $\mu(\mathfrak{E}_1') = \mu(\mathfrak{E}_1)$, *i.e.* $\mathfrak{E}_1$ is the maximal destabilizing Higgs subsheaf of $\mathfrak{E}$. Q.e.d.

1.4.1 Properties of Harder-Narasimhan filtration of Higgs sheaves, Part I

Using the notations (1.3), let $\mu_{\min}(\mathfrak{E}) = \mu_m$ and let $\mu_{\max}(\mathfrak{E}) = \mu_1$.

Proposition 1.4.6. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be torsion-free Higgs sheaves on X.*

a) If $\mu_{\min}(\mathfrak{E}_1) > \mu_{\max}(\mathfrak{E}_2)$ then $\mathrm{Hom}(\mathfrak{E}_1, \mathfrak{E}_2) = \{0\}$.

b) If there exists a surjective morphism $e\colon \mathfrak{E}_1 \twoheadrightarrow \mathfrak{E}_2$ then $\mu_{\min}(\mathfrak{E}_1) \leq \mu_{\min}(\mathfrak{E}_2)$.

c) If there exists an injective morphism $m\colon \mathfrak{E}_1 \rightarrowtail \mathfrak{E}_2$ then $\mu_{\max}(\mathfrak{E}_1) \leq \mu_{\max}(\mathfrak{E}_2)$.

d) $\mu_{\min}(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \min\{\mu_{\min}(\mathfrak{E}_1), \mu_{\min}(\mathfrak{E}_2)\}$, $\mu_{\max}(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \min\{\mu_{\max}(\mathfrak{E}_1), \mu_{\max}(\mathfrak{E}_2)\}$.

In the following proofs, $\{0\} = \mathfrak{E}_{k,0} \subsetneqq \mathfrak{E}_{k,1} \subsetneqq \dots \subsetneqq \mathfrak{E}_{k,m_k-1} \subsetneqq \mathfrak{E}_{k,m_k} = \mathfrak{E}_k$ is the HN-filtration of $\mathfrak{E}_k$, where $k \in \{1, 2\}$.

Proof. (a). Suppose that there exists $0 \neq f \in \mathrm{Hom}(\mathfrak{E}_1, \mathfrak{E}_2)$, let i be the minimal index such that $f(\mathfrak{E}_{1,i}) \neq 0$ and let j be the minimal index such that $f(\mathfrak{E}_{1,i}) \subseteq \mathfrak{E}_{2,j}$. Then there is a non-trivial morphism $\widetilde{f}\colon \mathfrak{E}_{1,i}/\mathfrak{E}_{1,i-1} \to \mathfrak{E}_{2,j}/\mathfrak{E}_{2,j-1}$, but

$$\mu(\mathfrak{E}_{1,i}/\mathfrak{E}_{1,i-1}) \geq \mu_{\min}(\mathfrak{E}_1) > \mu_{\max}(\mathfrak{E}_2) \geq \mu(\mathfrak{E}_{2,j}/\mathfrak{E}_{2,j-1})$$

and this is absurd by item a.

(b). Considering the following diagram

$$\begin{tikzcd}
\mathfrak{E}_1 \arrow[r, "e"] \arrow[dr, "e_0"'] & \mathfrak{E}_2 \arrow[r] \arrow[d] & 0 \\
& \mathfrak{E}_2/\mathfrak{E}_{2,m_2-1} \arrow[d] & \\
& 0 &
\end{tikzcd}$$

where the row and column are right exact and the triangle commutes. Since e is a surjective morphism, e_0 is not the zero morphism; by Proposition item a:

$$\mu_{\min}(\mathfrak{E}_1) \leq \mu_{\max}(\mathfrak{E}_2/\mathfrak{E}_{2,m_2-1}) = \mu(\mathfrak{E}_2/\mathfrak{E}_{2,m_2-1}) \leq \mu_{\min}(\mathfrak{E}_2).$$

(c). Let j be the minimal index such that $m(\mathfrak{E}_{1,1}) \subseteq \mathfrak{E}_{2,j}$, by hypothesis we have a non-zero morphism $\widetilde{m} : \mathfrak{E}_{1,1} \to \mathfrak{E}_{2,j}/\mathfrak{E}_{2,j-1}$, then by Proposition 1.3.4.a:

$$\mu_{\max}(\mathfrak{E}_1) = \mu(\mathfrak{E}_{1,1}) \leq \mu(\mathfrak{E}_{2,j}/\mathfrak{E}_{2,j-1}) \leq \mu_{\max}(\mathfrak{E}_2).$$

(d). By items c and b:

$$\mu_{\min}(\mathfrak{E}_1 \oplus \mathfrak{E}_2) \leq \min\{\mu_{\min}(\mathfrak{E}_1), \mu_{\min}(\mathfrak{E}_2)\},$$
$$\mu_{\max}(\mathfrak{E}_1 \oplus \mathfrak{E}_2) \leq \min\{\mu_{\max}(\mathfrak{E}_1), \mu_{\max}(\mathfrak{E}_2)\}.$$

By the proof of Theorem 1.4.2 there exists a positive rank Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ such that $\mu(\mathfrak{F}) = \mu_{\min}(\mathfrak{E}_1 \oplus \mathfrak{E}_2)$ and is semistable. Since $\mathfrak{F}$ is the maximal destabilizing Higgs subsheaf of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$, we have:

$$\min\{\mu_{\min}(\mathfrak{E}_1), \mu_{\min}(\mathfrak{E}_2)\} \leq \mu(\mathfrak{F}),$$

i.e. we prove the first equality. Analogously, by the proof of Theorem 1.4.2 there exists a torsion-free Higgs quotient $\mathfrak{Q}$ of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ for which $\mu(\mathfrak{Q}) = \mu_{\max}(\mathfrak{E}_1 \oplus \mathfrak{E}_2)$ and it is semistable. Since $\mathfrak{Q}$ is the maximal destabilizing torsion-free Higgs quotient of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$, we have:

$$\min\{\mu_{\max}(\mathfrak{E}_1), \mu_{\max}(\mathfrak{E}_2)\} \leq \mu(\mathfrak{Q})$$

i.e. we prove the second equality. Q.e.d.

1.4.2 Harder-Narasimhan polygon for Higgs sheaves

Using the definition 1.4.1 of the HN-filtration, we consider the points $(\mathrm{rank}(\mathfrak{E}_i), \deg(\mathfrak{E}_i)) \in \mathbb{R}^2$ and we denote them P_i, where $i \in \{0, \ldots, m\}$.

Definition 1.4.7. The convex hull of the set $\{P_i \in \mathbb{R}^2\}_{i \in \{0,\ldots,m\}}$ is called the *HN-polygon of* $\mathfrak{E}$.

Remark 1.4.8. By construction, the slope of the line passing through the points P_{i-1} and P_i is μ_i, for any $i \in \{1, \ldots, m\}$. The slope of the line passing through the points P_0 and P_m is $\mu(\mathfrak{E})$. Since $\mu_1 > \ldots > \mu_m$, the line segments $\overline{P_0 P_m}, \overline{P_{i-1} P_i}$ are the sides of the HN-polygon of $\mathfrak{E}$. $\Diamond$

Theorem 1.4.9 (**Maximal Property of HN-Polygon**, cfr. [60, Theorem 2])**.** *Let $\mathfrak{F}$ be a positive rank Higgs subsheaf of $\mathfrak{E}$. The point $(\mathrm{rank}(\mathfrak{F}), \deg(\mathfrak{F})) \in \mathbb{R}^2$ lies either on or below the HN-polygon of $\mathfrak{E}$. As a consequence, any polygon associated to a filtration of $\mathfrak{E}$ in Higgs subsheaves is dominated*[5] *by the HN-polygon.*

[5]Let $\{0\} = \mathfrak{F}_0 \subsetneqq \mathfrak{F}_1 \subsetneqq \cdots \subsetneqq \mathfrak{F}_{m-1} \subsetneqq \mathfrak{F}_m = \mathfrak{E}$ be a filtration of $\mathfrak{E}$ in Higgs subsheaves. We posit $Q_j = (\mathrm{rank}(\mathfrak{F}_j), \deg(\mathfrak{F}_j)) \in \mathbb{R}^2$ for any $j \in \{1, \ldots, m\}$. Since any Q_j is on or below the HN-polygon of $\mathfrak{E}$, we say that the polygon determined by Q_j's is *dominated* by HN-polygon.

Proof. Let $\mathfrak{E}$ be semistable, in particular this happens if $\operatorname{rank}(\mathfrak{E}) = r = 1$. By definition, for any positive rank Higgs subsheaf $\mathfrak{F}$ we have $\mu(\mathfrak{F}) \leq \mu(\mathfrak{E})$, in other words we prove the claim. Otherwise, let $\mathfrak{E}$ be unstable; in particular $r \geq 2$. By induction, we can assume that the statement holds for torsion-free Higgs sheaves of rank at most $r-1$. By the proof of Theorem 1.4.2 we have $\mu(\mathfrak{F}) \leq \mu(\mathfrak{E}_1)$, where $\mathfrak{E}_1$ is the maximal destabilizing Higgs subsheaf of $\mathfrak{E}$. If $\mu(\mathfrak{F}) = \mu(\mathfrak{E}_1)$ then the point in $\mathbb{R}^2$ associated to $\mathfrak{F}$ lies on the HN-polygon of $\mathfrak{E}$ since, by the previous reasoning, $0 < \operatorname{rank}(\mathfrak{F}) \leq \operatorname{rank}(\mathfrak{E}_1)$. From now on, let $\mu(\mathfrak{F}) < \mu(\mathfrak{E}_1)$. Considering the torsion-free Higgs sheaves $\mathfrak{A} = \mathfrak{F} \cap \mathfrak{E}_1$ and $\mathfrak{B} = \mathfrak{F} + \mathfrak{E}_1$; since $\deg(\mathfrak{A}) \leq \mu_1 \operatorname{rank}(\mathfrak{A})$ and $\deg(\mathfrak{A}) + \deg(\mathfrak{B}) \geq \deg(\mathfrak{E}_1) + \deg(\mathfrak{F})$ we have

$$\deg(\mathfrak{B}) \geq \mu_1(\operatorname{rank}(\mathfrak{E}_1) - \operatorname{rank}(\mathfrak{A})) + \mu(\mathfrak{F}) \operatorname{rank}(\mathfrak{F}).$$

If $\operatorname{rank}(\mathfrak{A}) = \operatorname{rank}(\mathfrak{E}_1)$ then $\mathfrak{F} \supseteq \mathfrak{E}_1$ and therefore $\mathfrak{F}/\mathfrak{E}_1$ is a Higgs subsheaf of $\mathfrak{E}/\mathfrak{E}_1$. By inductive hypothesis, the point in $\mathbb{R}^2$ associated to $\mathfrak{F}/\mathfrak{E}_1$ lies either on or below the HN-polygon of $\mathfrak{E}/\mathfrak{E}_1$; in particular the coordinates of this point are the difference of the coordinates of the points associated to $\mathfrak{F}$ and $\mathfrak{E}_1$. In other words, the point associated to $\mathfrak{F}$ lies either on or below the HN-polygon of $\mathfrak{E}$. If $\operatorname{rank}(\mathfrak{A}) < \operatorname{rank}(\mathfrak{E}_1)$ then we have

$$\deg(\mathfrak{B}) \geq \mu_1(\operatorname{rank}(\mathfrak{E}_1) - \operatorname{rank}(\mathfrak{A})) + \mu(\mathfrak{F}) \operatorname{rank}(\mathfrak{F}) > \mu(\mathfrak{F})(\operatorname{rank}(\mathfrak{E}_1) - \operatorname{rank}(\mathfrak{A}) + \operatorname{rank}(\mathfrak{F}))$$
$$\mu(\mathfrak{B}) = \frac{\deg(\mathfrak{B})}{\operatorname{rank}(\mathfrak{E}_1) - \operatorname{rank}(\mathfrak{A}) + \operatorname{rank}(\mathfrak{F})} > \mu(\mathfrak{F}).$$

Since $\mathfrak{B} \supseteq \mathfrak{E}_1$, $\mathfrak{B}/\mathfrak{E}_1$ is a Higgs subsheaf of $\mathfrak{E}/\mathfrak{E}_1$; repeating the previous reasoning, the point associated to $\mathfrak{B}$ lies either on or below the HN-polygon of $\mathfrak{E}$. Still $\operatorname{rank}(\mathfrak{F}) \leq \operatorname{rank}(\mathfrak{B})$, by inductive hypothesis and the last statement, the point associated to $\mathfrak{F}$ lies either on or below the HN-polygon of $\mathfrak{E}$. Q.e.d.

1.5 Jordan-Hölder filtrations for Higgs sheaves

Any semistable Higgs sheaf admits a filtration whose successive Higgs quotient sheaves are stable.

Indeed, let $\mathfrak{E}$ be a semistable Higgs sheaf. If it is stable then we finish, otherwise there exists a Higgs subsheaf $\mathfrak{E}_1$ of $\mathfrak{E}$ such that $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E})$ and its rank is minimal. The Higgs sheaf $\mathfrak{E}_1$ is stable and the quotient $\mathfrak{E}/\mathfrak{E}_1$ is semistable. Iterating this process at the end we obtain a filtration

$$\{0\} = \mathfrak{E}_0 \subsetneqq \mathfrak{E}_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}_{m-1} \subsetneqq \mathfrak{E}_m = \mathfrak{E},$$

where the successive Higgs quotient sheaves $\mathfrak{E}_i/\mathfrak{E}_{i-1}$ are stable for any $i \in \{1, \ldots, m\}$, and $\mu(\mathfrak{E}_i) = \mu(\mathfrak{E})$.

Definition 1.5.1. The previous filtration is called a *Jordan-Hölder filtration of* $\mathfrak{E}$ (*JH-filtration*, for short).

This JH-filtration of $\mathfrak{E}$, while it is not unique, satisfies the following theorems.

Theorem 1.5.2. *The JH-filtrations of* $\mathfrak{E}$ *have the same length.*

Proof. Let $\mathfrak{E}$ be semistable but not stable, and let

$$\{0\} = \mathfrak{E}^1_0 \subsetneqq \mathfrak{E}^1_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}^1_{m-1} \subsetneqq \mathfrak{E}^1_t = \mathfrak{E} \tag{1.4}$$

$$\{0\} = \mathfrak{E}^2_0 \subsetneqq \mathfrak{E}^2_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}^2_{l-1} \subsetneqq \mathfrak{E}^2_s = \mathfrak{E} \tag{1.5}$$

be two different JH-filtrations of $\mathfrak{E}$. By absurd let $s > t$. Let i be the minimum index such that $\mathfrak{E}^1_1 \subseteq \mathfrak{E}^2_i$ and $\mathfrak{E}^1_1 \nsubseteq \mathfrak{E}^2_{i-1}$. Then the composed morphism $\mathfrak{E}^1_1 \hookrightarrow \mathfrak{E}^2_i \to \mathfrak{E}^2_i/\mathfrak{E}^2_{i-1}$ is non-zero between stable Higgs sheaves of the same slope, by Proposition 1.3.4.e it is an isomorphism. Hence $\mathfrak{E}^2_i = \mathfrak{E}^2_{i-1} \oplus \mathfrak{E}^1_1$ and we have the following JH-filtration of $\mathfrak{E}$

$$\{0\} \subsetneqq \mathfrak{E}^1_1 \subsetneqq \mathfrak{E}^2_1 \oplus \mathfrak{E}^1_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}^2_{i-1} \oplus \mathfrak{E}^1_1 \subsetneqq \mathfrak{E}^2_{i+1} \subsetneqq \ldots \subsetneqq \mathfrak{E}^2_{l-1} \subsetneqq \mathfrak{E}^2_l = \mathfrak{E}. \tag{1.6}$$

Since this JH-filtration has the first term equal to the first term of the JH-filtration (1.4), they induce JH-filtrations of $\mathfrak{E}/\mathfrak{E}^1_1$, of length $s-1$ and $t-1$, respectively. Repeating this reasoning $t-1$ times, we find a JH-filtration of $\mathfrak{E}/\mathfrak{E}^2_{l-1}$ of length $s-t > 0$. This means that $\mathfrak{E}^2_{l-1} \subsetneqq \mathfrak{E}^1_{m-1} \subsetneqq \mathfrak{E}^1_m = \mathfrak{E}$ and by hypothesis $\mu(\mathfrak{E}/\mathfrak{E}^2_{l-1}) = \mu(\mathfrak{E}/\mathfrak{E}^1_{m-1})$, *i.e.* $\mathfrak{E}/\mathfrak{E}^2_{l-1}$ has a proper Higgs subsheaf of the same slope. This contradicts the stability condition of $\mathfrak{E}/\mathfrak{E}^2_{l-1}$. To avoid all this, we must have $s = t$. Q.e.d.

Theorem 1.5.3. *The graded Higgs sheaves* $\mathrm{Gr}(\mathfrak{E}) = \bigoplus_{i=1}^{s} \mathfrak{E}_i/\mathfrak{E}_{i-1}$ *of all JH-filtrations of* $\mathfrak{E}$ *are isomorphic.*

Remark 1.5.4. By Definition 1.3.3, $\mathrm{Gr}(\mathfrak{E})$ is a polystable Higgs sheaf. ♢

Proof. Using the same notation of the JH-filtrations (1.4) and (1.5), let $k = s = t$. If $k = 1$ there is nothing to prove. Let $k \geq 2$, then the graded Higgs sheaves associated to JH-filtrations (1.5) and (1.6) are isomorphic by construction. The JH-filtrations (1.4) and (1.6) induce JH-filtrations of $\mathfrak{E}/\mathfrak{E}^1_1$ of length $k-1$; by inductive hypothesis, their graded Higgs sheaves are isomorphic. We have

$$\mathrm{Gr}\left(\mathfrak{E}^2_s\right) \cong \mathrm{Gr}\left(\mathfrak{E}^2_l\right) \cong \mathrm{Gr}\left(\mathfrak{E}^2_l/\mathfrak{E}^1_1\right) \oplus \mathfrak{E}^1_1 \cong \mathrm{Gr}\left(\mathfrak{E}^1_t/\mathfrak{E}^1_1\right) \oplus \mathfrak{E}^1_1 \cong \mathrm{Gr}\left(\mathfrak{E}^1_t\right).$$

Q.e.d.

Chapter 2

Tensor product of semistable Higgs bundles

In [2, Section 6] Balaji and Parameswaran, using G.I.T. techniques, have proved that the tensor product of semistable Higgs bundles over smooth projective curves over $\mathbb{K}$ is semistable as well, also when $\mathbb{K}$ has positive characteristic. On the other hand, using techniques of complex geometry, Simpson has proved the same result on smooth complex projective varieties ([62, Corollary 3.8]). More in general, with regard to compact Kähler manifolds, Biswas and Schumacher ([5, Proposition 4.5]) and Holguín Cardona ([34, Theorem 5.4]) have proved that the tensor product of semistable Higgs shaves is semistable as well.

Here we prove a "Lefschetz principle"-type theorem for semistable Higgs sheaves which allows us to give another proof of the semistability of the tensor products of semistable Higgs sheaves over smooth projective varieties, defined over an algebraically closed field $\mathbb{K}$ of characteristic 0.

2.1 Projective varieties and base change

We fix the following definitions.

Definition 2.1.1. Let $\mathbb{F}$ be a field and let $\mathbb{F}'$ be an its extension field.

a) ([23, Definition at page 551]) We say $\mathbb{F}'$ *algebraic separable* if it is algebraic and the minimal polynomial p of any element of $\mathbb{F}'$ is *coprime* with its derivative Dp.

b) ([1, tag 030E]) A collection of elements $\{x_i\}_{i\in I}$ of $\mathbb{F}'$ is called *algebraically independent* over $\mathbb{F}$ if the map
$$\mathbb{F}[X_i; i \in I] \to \mathbb{F}'$$
which maps X_i to x_i is injective.

c) ([1, tag 030E]) A *transcendence basis of* $\mathbb{F}'$ *over* $\mathbb{F}$ is a collection of elements $\{x_i\}_{i\in I}$ which are algebraically independent over $\mathbb{F}$ and such that $\mathbb{F}'$ is an algebraic extension of $\mathbb{F}(x_i; i \in I)$.

d) ([30, Definition at page 27]) We say $\mathbb{F}'$ is *separably generated* over $\mathbb{F}$ if there exists a transcendence basis $\{x_i \in \mathbb{F}'\}_{i\in I}$ such that $\mathbb{F}'$ is an algebraic separable extension of $\mathbb{F}(x_i; i \in I)$.

e) ([1, tag 030O]) We say $\mathbb{F}'$ *separable* over $\mathbb{F}$ if for any extension $\mathbb{F} \subseteq \mathbb{F}'' \subseteq \mathbb{F}'$, with $\mathbb{F}''$ finitely generated over $\mathbb{F}$, $\mathbb{F}''$ is separably generated over $\mathbb{F}$.

f) ([1, tag 030Y]) We say $\mathbb{F}$ *perfect* if any its extension is separable.

Remark 2.1.2.

a) Any field extension has a transcendence basis ([23, First Theorem in Section 14.9]).

b) Any separably generated field extension is separable. ([1, tag 030X]).

c) The algebraic extensions of a characteristic 0 field are separable, hence any field of characteristic 0 is perfect.

d) A field of characteristic 0 is separably closed if and only if it is algebraically closed. $\diamondsuit$

From now on, let X be a smooth projective variety defined over $\mathbb{K}$, an algebraically closed field of characteristic 0, let $\mathbb{F}$ be an extension field of $\mathbb{K}$ and let $X_{\mathbb{F}} = X \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{F})$. By [1, tags 01WF and 020J], [31, Proposition III.10.1.(b)] and [7, Lemma 8.3.3.i] $X_{\mathbb{F}}$ is an irreducible smooth projective scheme of finite type over $\mathbb{F}$. Moreover, we have the following lemma

Lemma 2.1.3. *$X_{\mathbb{F}}$ is a smooth projective variety.*

Proof. X is a reduced scheme *i.e.* for any open subset U of X, $\mathcal{O}_X(U)$ is a reduced $\mathbb{K}$-algebra, by [1, tags 030S and 030V] $\mathcal{O}_X(U)$ is a *geometrically reduced* $\mathbb{K}$*-algebra*. Since $\mathbb{K}$ is a perfect field, then $\mathbb{F}$ is a separable extension of $\mathbb{K}$, hence $\mathcal{O}_X(U) \otimes_{\mathbb{K}} \mathbb{F}$ is a reduced $\mathbb{F}$-algebra ([1, tag 030U]). Thus, $\{f^{-1}(U) = \mathrm{Spec}\,(\mathcal{O}_X(U) \otimes_{\mathbb{K}} \mathbb{F})\}_{U\subseteq X \text{ open}}$ is an affine open covering of $X_{\mathbb{F}}$ given by reduced subschemes, hence $X_{\mathbb{F}}$ is a reduced scheme ([46, Proposition 2.4.2.b]). Here f is defined by the following Cartesian diagram

$$\begin{tikzcd} X_{\mathbb{F}} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r] & \mathrm{Spec}(\mathbb{K}) \end{tikzcd} .$$

We infer that $X_{\mathbb{F}}$ is an integral scheme by [30, Proposition II.3.1]. Q.e.d.

Definition 2.1.4 (cfr. [67, Definition 2.4])**.** A *fpqc*[1] morphism $\phi\colon S \to T$ of schemes is a faithfully flat morphism[2] for which there exists an affine open covering $\{T_i\}_{i\in I}$ of T, such that each T_i is the image of a quasi-compact open subset of S.

Proposition 2.1.5. *The canonical morphism $f\colon X_{\mathbb{F}} \to X$ is fpqc.*

Proof. Since $X_{\mathbb{F}}$ is a closed subscheme of $\mathbb{P}^N_{\mathbb{F}}$ ([7, Proposition 7.3.13]) for some $N \in \mathbb{N}_{\geq 1}$, $X_{\mathbb{F}}$ is quasi-compact. Thus we can consider a finite affine open covering $\{f^{-1}(U_i)\}_{i\in\{1,\ldots,m\}}$ of $X_{\mathbb{F}}$, where each U_i is an open subscheme of X. By [7, Proposition 4.4.1.iii and Corollary 7.2.7], f is faithfully flat, $f^{-1}(U_i) = \mathrm{Spec}\,(\mathcal{O}_X(U_i) \otimes_{\mathbb{K}} \mathbb{F})$ for any $i \in \{1,\ldots,m\}$ and these are quasi-compact topological spaces ([30, Exercise II.2.13.b]). In other words $U_i = f\,(\mathrm{Spec}\,(\mathcal{O}_X(U_i) \otimes_{\mathbb{K}} \mathbb{F}))$ for any $i \in \{1,\ldots,m\}$, *i.e.* the claim holds. Q.e.d.

2.2 Higgs sheaves, their semistability and base change

We begin to prove the following lemma.

Lemma 2.2.1. *Let $\mathfrak{E} = (E,\varphi)$ be a Higgs bundle over X. Then there exist an algebraically closed subfield $\mathbb{K}_0$ of $\mathbb{K}$, a variety $X_{\mathbb{K}_0}$ over $\mathbb{K}_0$ and a Higgs bundle $\mathfrak{E}_0$ over $X_{\mathbb{K}_0}$, such that $\mathbb{K}_0$ is isomorphic to a subfield of $\mathbb{C}$, the following diagram is Cartesian*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X_{\mathbb{K}_0} \arrow[d] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}\,(\mathbb{K}_0) \end{tikzcd}$$

and $\mathfrak{E} = f^*\mathfrak{E}_0$.

[1] In French, "*fidèlement plat et quasi-compact*".
[2] In other words, f is a surjective flat morphism of schemes ([67, Definition 1.10]).

Proof. By definition $X = \operatorname{Proj}\left(\dfrac{\mathbb{K}\left[x_0,\dots,x_n\right]}{J}\right)$. To give a Higgs bundle over X is equivalent to given a triple $\{U_i,\lambda_{ij},\varphi_i\}_{i,j\in I}$ where

- I is a finite set of indexes, because X is quasi-compact as topological space;
- U_i's are open affine subsets of X which recovers it;
- $\lambda_{ij}\colon \mathcal{O}_{U_{ij}}^{\oplus r} \to \mathcal{O}_{U_{ij}}^{\oplus r}$'s are the transition functions of E, where $U_{ij} = U_i \cap U_j$ are affine open subsets of X;
- $\varphi_i = \varphi\left(U_i\right) : \mathcal{E}\left(U_i\right) \to \left(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X\right)\left(U_i\right) = \mathcal{E}\left(U_i\right) \otimes_{\mathcal{O}_X(U_i)} \Omega^1_X\left(U_i\right)$, this last equality holds because $\mathcal{E}$ and Ω^1_X are coherent $\mathcal{O}_X$-modules and the U_i's are open affine subsets of X.

By [30, Proposition II.5.2.(a)], the morphisms λ_{ij}'s and φ_i's correspond to morphisms $\widetilde{\lambda_{ij}}$'s and $\widetilde{\varphi}_i$'s of opportune coherent modules.

Since all these are modules of finite type over $\mathbb{K}$, we can consider the set S of all coefficient of the polynomial which generate the ideals of these modules. By Noetherianity of these modules, S is a finite set and let $\alpha_1,\dots,\alpha_p,\tau_1,\dots,\tau_q$ its elements, where each α_i is algebraic on $\mathbb{Q}$ and the τ_j's are algebraically independent on $\mathbb{Q}$ (Definition 2.1.1.b); if one of these types of elements does not occur we have either $p = 0$ or $q = 0$. Let $\mathbb{K}_0$ be the algebraically closure of the field generated by S over $\mathbb{Q}$; $\mathbb{K}_0$ is a subfield of $\mathbb{C}$. Indeed, by definition

$$\mathbb{K}_0 = \overline{Q\left(\mathbb{Q}(\alpha_1,\dots,\alpha_p)[\tau_1,\dots,\tau_q]\right)}^{alg}$$

where $Q(\_)$ is the quotient field of $\_$. Let $t_1,\dots,t_q \in \mathbb{C}$ be transcendental numbers algebraically independent on $\mathbb{Q}$, then

$$\mathbb{K}_0 \cong \overline{Q\left(\mathbb{Q}(\alpha_1,\dots,\alpha_p)[t_1,\dots,t_q]\right)}^{alg} \subsetneq \mathbb{C}.$$

Let J_0 be the ideal of $\mathbb{K}_0\left[x_0,\dots,x_n\right]$ generated by the element of J view as elements of this ring, let $X_{\mathbb{K}_0} = \operatorname{Proj}\left(\dfrac{\mathbb{K}_0\left[x_0,\dots,x_n\right]}{J_0}\right)$. By construction we have the Cartesian diagram of the claim.

Using the same reasoning, we define a triple $\left\{U_i^0,\lambda_{ij}^0,\varphi_i^0\right\}_{i,j\in I}$ which determines a Higgs bundle $\mathfrak{E}_0 = (E_0,\varphi_0)$ over $X_{\mathbb{K}_0}$ such that $f^*\mathfrak{E}_0 = \mathfrak{E}$. Q.e.d.

We explain all details of the proof of [43, Proposition 1].

From now on, ξ is the generic point of X and let $\mathcal{E}$ be a torsion-free coherent sheaf on X.

Lemma 2.2.2. *Let $\mathbb{W}$ be a vector subspace of $\mathcal{E}(\xi) \cong \mathcal{E}_\xi$. Then there exists a unique saturated torsion-free coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that $\mathbb{W} = \mathcal{F}(\xi)$.*

Proof. By Nakayama's Lemma, there exists a $\mathcal{O}_{X,\xi}$-submodule $\mathcal{W}_\xi$ of $\mathcal{E}_\xi$ such that $\mathcal{W}_\xi \cong \mathbb{W}$. As in [43, Proposition 1], we construct a torsion-free sheaf on X as it follows: let $\mathcal{U}$ be the collection of all open subset of X such that $\mathcal{E}_{|U}$ is a quotient of a locally free sheaf, this is a *cofinal system* to collection $\mathbf{Op}(X)$ of all open subsets of X therefore we will compute the direct limits over $\mathcal{U}$ and not over $\mathbf{Op}(X)$.

- $\forall U \in \mathcal{U}$ open, we let $\mathcal{W}(U) = \left(\rho_\xi^U\right)^{-1}(\mathcal{W}_\xi)$, where $\rho_\xi^U : \mathcal{O}_X(U)^{\oplus r} \to \mathcal{E}_\xi$ is the canonical morphism;
- $\forall x \in X$, we let $\mathcal{W}_x = \varinjlim_{x \in U} \mathcal{W}(U)$;
- $\forall x \in X$, $\mathcal{F}_x = \mathcal{E}_x \cap \mathcal{W}_x$.

Let $\mathcal{B}_\xi = \left\{ \underline{e}_1^\xi, \dots, \underline{e}_m^\xi \right\}$ be a basis of $\mathcal{W}_\xi$. By construction, for any $U \subseteq X$ open, $\mathcal{S}(U) = \left(\rho_\xi^U\right)^{-1}(\mathcal{B}_\xi)$ is a system of generators of $\mathcal{W}(U)$. Passing to inductive limit, we obtain a system of generators $\mathcal{S}_x$ of $\mathcal{W}_x$ for any $x \in X$, hence we get a system of generators $\mathcal{S}(x)$ of $\mathcal{W}(x) = \mathcal{W}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. Using the elements of $\mathcal{S}(x)$ we construct a finite basis $\mathcal{B}(x) = \left\{ \underline{e}_1(x), \dots, \underline{e}_{m_x}(x) \right\}$ of $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. By Nakayama's Lemma, $\mathcal{B}(x)$ gives rise to a finite system of generators $\mathcal{B}_x$ of $\mathcal{F}_x$. Using the Geometric Version of Nakayama's Lemma, for any $x \in X$ there exists $U \subseteq X$ open and affine such that $x \in U$ and $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-submodule of finite type of $\mathcal{E}(U)$. Since the $\mathcal{O}_X(U)$'s are Noetherian modules, then $\mathcal{F}(U)$ are coherent $\mathcal{O}_X(U)$-modules, *i.e.* $\mathcal{F}$ is a coherent torsion-free $\mathcal{O}_X$-submodule of $\mathcal{E}$.

Let $\mathcal{F}'$ another coherent torsion-free $\mathcal{O}_X$-submodule of $\mathcal{E}$ such that $\mathcal{F}'_\xi = \mathcal{W}_\xi$. By the previous construction

$$\forall x \in X,\ \mathcal{F}'_x \subseteq \mathcal{F}_x$$

i.e. $\mathcal{F}' \subseteq \mathcal{F}$. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{Q} = \mathrm{coker}(\mathcal{F} \to \mathcal{E}) \arrow[r] & 0\,, \\
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[u, hook, "i"] & \mathcal{E} \arrow[r] \arrow[u, equal] & \mathcal{Q}' = \mathrm{coker}\left(\mathcal{F}' \to \mathcal{E}\right) \arrow[r] \arrow[u, "q"'] & 0
\end{tikzcd}$$

it follows from the proof of [43, Proposition 1] that $\mathcal{Q}$ and $\mathcal{Q}'$ are coherent torsion-free $\mathcal{O}_X$-modules. By the universal property of cokernels of morphisms, there exists a unique morphism $q\colon \mathcal{Q}' \to \mathcal{Q}$ which makes commutative the diagram. By the Four Lemma q is an epimorphism. Since $q(\xi)\colon \mathcal{Q}'(\xi) \to \mathcal{Q}(\xi)$ is an isomorphism of $\kappa(\xi)$-vector spaces, by Nakayama's Lemma q_ξ is an isomorphism of $\mathcal{O}_{X,\xi}$-modules, and this implies that $\ker(q)_\xi = \{0\}$, *i.e.* $\ker(q)$ is a torsion subsheaf of $\mathcal{Q}'$. This is possible if and only if $\ker(q) = \underline{0}_X$ hence $\mathcal{Q}' \cong \mathcal{Q}$. Thus $\mathcal{F}' = \mathcal{F}$. Q.e.d.

Lemma 2.2.3. *Let $\mathfrak{E} = (\mathcal{E}, \varphi)$ be a Higgs sheaf on X. Consider the following Cartesian diagram*

$$\begin{tikzcd} X_{\mathbb{F}} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(\mathbb{F}) \arrow[r] & \operatorname{Spec}(\mathbb{K}) \end{tikzcd} .$$

If $\mathcal{F}$ is a subsheaf of $\mathcal{E}$ such that $\left(f^*\mathcal{F}, f^*\varphi_{|f^*\mathcal{F}}\right)$ is a Higgs subsheaf of $f^*\mathfrak{E}$, then $\left(\mathcal{F}, \varphi_{|\mathcal{F}}\right)$ is a Higgs subsheaf of $\mathfrak{E}$.

On $f^*\mathfrak{E}$ one defines the following Higgs field

$$f^*\mathfrak{E} \xrightarrow{f^*\varphi} f^*\mathfrak{E} \otimes_{\mathcal{O}_{X_{\mathbb{F}}}} f^*\Omega^1_X \xrightarrow{Id\otimes f^*} f^*\mathfrak{E} \otimes_{\mathcal{O}_{X_{\mathbb{F}}}} \Omega^1_{X_{\mathbb{F}}}$$

which is denoted, by abuse of notation, as $f^*\varphi$.

Proof. Recall that

$$\forall y \in X_{\mathbb{F}},\ (f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_{\mathbb{F}},y},\ \left(f^*\Omega^1_X\right)_y = \Omega^1_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_{\mathbb{F}},y}$$

where $x = f(y)$, one has

$$\forall y \in X_{\mathbb{F}},\ (f^*\mathcal{F})(y) \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(y),\ \left(f^*\Omega^1_X\right)(y) \cong \Omega^1_{X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(y).$$

Let $\{s_{i,x} \in \mathcal{F}_x\}_{i\in I_x}$ be a system of generators of $\mathcal{F}_x$ and let $\left\{e_{j,x} \in \Omega^1_{X,x}\right\}_{j\in J_x}$ be a basis of $\Omega^1_{X,x}$. Then $\left\{f^*_x s_{i,x} \otimes f^* e_{j,x} \in f^*\mathcal{F}_y \otimes_{\mathcal{O}_{X_{\mathbb{F}},y}} \Omega^1_{X_{\mathbb{F}},y}\right\}_{\substack{i\in I_x\\ j\in J_x}}$ is a system of generators of $(f^*\varphi_y)(f^*\mathcal{F}_y)$; however $\left\{f^*_x s_{i,x} \otimes f^* e_{j,x} \in f^*\mathcal{F}_y \otimes_{\mathcal{O}_{X_{\mathbb{F}},y}} f^*\Omega^1_{X,x}\right\}_{\substack{i\in I_x\\ j\in J_x}}$ is also a system of generators of $f^*\mathcal{F}_y \otimes_{\mathcal{O}_{X_{\mathbb{F}},y}} f^*\Omega^1_{X,x}$, thus $\operatorname{Im}\left(f^*\varphi_{|f^*\mathcal{F}}\right) \subseteq f^*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X\right)$ up to isomorphisms; in other words, $f^*\varphi_{|f^*\mathcal{F}}$ factorizes through the morphism $\psi\colon f^*\mathcal{F} \to f^*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X\right)$. Since by Proposition 2.1.5 $f\colon X_{\mathbb{F}} \to X$ is a fpqc morphism, let $\{U_i\}_{i\in\{1,\dots,m\}}$ be a finite affine open covering of X, this defines *descent data* ([1, tag 023B]) $\left\{f^*\mathcal{F}_{|f^{-1}(U_i)}, \mathrm{Id}_{f^*\mathcal{F}_{|f^{-1}(U_i\cap U_j)}}\right\}_{i,j\in\{1,\dots,m\}}$ and $\left\{f^*\left(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X\right)_{|f^{-1}(U_i)}, \mathrm{Id}_{f^*\left(\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^1_X\right)_{|f^{-1}(U_i\cap U_j)}}\right\}_{i,j\in\{1,\dots,m\}}$ and ψ is a *morphism of descent data* ([1, tag 023B]). By [1, tag 023T] there exists a unique morphism of sheaves

$\chi\colon \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X$ whose lift via f is ψ, by the previous construction $\varphi_{|\mathcal{F}}$ lifts to ψ hence $\varphi_{|\mathcal{F}} = \chi$ that is the claim holds. Q.e.d.

The following lemma extends [43, Proposition 3] to the Higgs bundles setting.

Lemma 2.2.4. *Let $\mathfrak{E} = (E, \varphi)$ a torsion-free Higgs sheaf over (X, H) and let $\mathbb{F}$ be an extension of $\mathbb{K}$. Consider the following Cartesian diagram*

$$\begin{tikzcd} X_{\mathbb{F}} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r] & \mathrm{Spec}(\mathbb{K}) \end{tikzcd}$$

then $f^*\mathfrak{E} = (f^*E, f^*\varphi)$ is a semistable Higgs sheaf if and only if $\mathfrak{E} = (E, \varphi)$ is semistable.

Proof. By base change f^*H is a polarization of $X_{\mathbb{F}}$ ([27, Proposition 4.6.13.iii]). As usual, for any torsion-free subsheaf $\mathcal{F}$ of $f^*\mathfrak{E}$, we set $\mu(\mathcal{F}) = \dfrac{c_1(\mathcal{F}) \cdot (f^*H)^{N-1}}{\mathrm{rank}(\mathcal{F})}$ with $\dim(X_{\mathbb{F}}) = N$.

If $\mathfrak{E}$ is unstable then there exists a torsion-free Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ such that $\mu(\mathfrak{F}) > \mu(\mathfrak{E})$, and without loss of generality, we can assume that $\mathcal{F}$ is reflexive. Since f is a flat morphism ([30, Proposition II.9.2.b]) then $f^*\mathcal{F}$ is also reflexive ([31, Proposition 1.8]) hence $\mu(f^*\mathfrak{F}) > \mu(f^*\mathfrak{E})$ *i.e.* $f^*\mathfrak{E}$ is unstable.

If $f^*\mathfrak{E}$ is unstable then there exists a saturated torsion-free Higgs subsheaf $\mathfrak{F} = \left(\mathcal{F}, f^*\varphi_{|\mathcal{F}}\right)$ such that $\mu(\mathfrak{F}) > \mu(f^*\mathfrak{E})$. Let $\bar{\xi}$ be the generic point of $X_{\mathbb{F}}$, since

$$\kappa(\xi) \otimes_{\mathbb{K}} \mathbb{F} = \mathcal{O}_{X,\xi} \otimes_{\mathbb{K}} \mathbb{F} \cong \mathcal{O}_{X_{\mathbb{F}},\bar{\xi}} = \kappa\left(\bar{\xi}\right).$$

Therefore up to isomorphism $\mathcal{F}\left(\bar{\xi}\right) \subseteq E\left(\bar{\xi}\right) = E(\xi) \otimes_{\mathbb{K}} \mathbb{F}$. Let $\{b_1, \ldots, b_s\}$ be a basis of $\mathcal{F}\left(\bar{\xi}\right)$, we can write

$$\forall i \in \{1, \ldots, s\},\ b_i = \sum_{j=1}^{r} a_i^j e_j$$

where $a_i^j \in \mathbb{F}$, $a_i^j e_j$ is $a_i^j \otimes e_j$ and $\{e_1, \ldots, e_r\}$ is a basis of $E(\xi)$. Let $\mathbb{F}_0$ be the extension of $\mathbb{K}$ generated by the a_i^j's and let $X_{\mathbb{F}_0} = X \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{F}_0)$; then $\{b_1, \ldots, b_s\}$ spans a vector subspace $\mathcal{F}_0(\xi_0) \subseteq E(\xi) \otimes_{\kappa(\xi)} \kappa(\xi_0)$, where ξ_0 is the generic point of $X_{\mathbb{F}_0}$. Consider the following commutative diagram

$$\begin{tikzcd} X_{\mathbb{F}} \arrow[r, "g_1"'] \arrow[rr, bend left, "f"] \arrow[d] & X_{\mathbb{F}_0} \arrow[r, "g_2"'] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r] & \mathrm{Spec}(\mathbb{F}_0) \arrow[r] & \mathrm{Spec}(\mathbb{K}) \end{tikzcd}$$

since

a) by Lemma 2.2.2 there exists a unique saturated torsion-free coherent subsheaf $\mathcal{F}_0$ of $g_2^*\mathcal{E}$ whose generic fibre is $\mathcal{F}_0\left(\xi_0\right)$,

b) $g_1^*\mathcal{F}_0 = \mathcal{F}$ and $\mu\left(\mathcal{F}_0\right) = \mu(\mathcal{F}) > \mu\left(f^*\mathfrak{E}\right) = \mu\left(g_2^*\mathfrak{E}\right)$,

c) by the previous point and by Lemma 2.2.3 $\left(\mathcal{F}_0, g_2^*\varphi_{|\mathcal{F}_0}\right)$ is a torsion-free Higgs subsheaf of $g_2^*\mathfrak{E}$,

we are reduced to proving the assert when $\mathbb{F}$ is a finitely generated extension of $\mathbb{K}$. Let $\{\alpha_1, \ldots, \alpha_p, \tau_1, \ldots, \tau_q \in \mathbb{F}_0\} \subseteq \left\{a_i^j \in \mathbb{F}_0\right\}$ a subset which is maximal algebraically independent; we can consider the chain of fields extension

$$\mathbb{K} \subseteq \mathbb{K}_0 = \mathbb{K}\left(\alpha_1 \ldots, \alpha_p\right) \subseteq \mathbb{K}_1 \subseteq \ldots \subseteq \mathbb{K}_{q-1} \subseteq \mathbb{F}_0$$

such that

- $\mathbb{K}_0$ is an algebraic extension of $\mathbb{K}$ of finite degree,
- $\operatorname{degtr}_{\mathbb{K}_{h-1}} \mathbb{K}_h = 1$ for any $h \in \{1, \ldots, q\}$, where we set $\mathbb{K}_h = \mathbb{K}_0\left(\tau_1, \ldots, \tau_h\right)$ and $\mathbb{K}_q = \mathbb{F}_0$.

By all this, the morphism g_2 can be split as following

$$\begin{tikzcd}
X_q \arrow[r, "h_q"'] \arrow[d] \arrow[rrrr, bend left, "g_2"] & X_{q-1} \arrow[r, "h_{q_1}"'] \arrow[d] & \ldots \arrow[r, "h_1"'] & X_0 \arrow[r, "h_0"'] \arrow[d] & X_{\mathbb{K}} \arrow[d] \\
\operatorname{Spec}\left(\mathbb{K}_q\right) \arrow[r] & \operatorname{Spec}\left(\mathbb{K}_{q-1}\right) \arrow[r] & \ldots \arrow[r] & \operatorname{Spec}\left(\mathbb{K}_0\right) \arrow[r] & \operatorname{Spec}(\mathbb{K})
\end{tikzcd}$$

where for any $h \in \{1, \ldots, q\}$, $X_h = X_{\mathbb{K}} \times_{\operatorname{Spec}(\mathbb{K})} \operatorname{Spec}\left(\mathbb{K}_h\right)$. Let G_q be the group of $\left(\mathbb{K}_{q-1}\left(\xi_1\right)\right)$-automorphism of $\mathbb{K}_q\left(\xi_0\right)$ generated by translation $\tau_q \mapsto \tau_q + v$ with $v \in \mathbb{K}_{q-1}$, with ξ_1 the generic point of $X_{\mathbb{K}_{q-1}}$. Each $\sigma_q \in G_q$ induces an automorphism $\widetilde{\sigma_q}$ of X_q over X_{q-1} such that $\widetilde{\sigma_q}^*\mathcal{F}_0\left(\xi_0\right) = \mathcal{F}_0\left(\xi_0\right)$ *i.e.* $\widetilde{\sigma_q}^*\mathcal{F}_0 = \mathcal{F}_0$ (Lemma 2.2.2). By [43, Lemma at page 98] and by [8, Theorem II.8.1.i] $\mathcal{F}_0\left(\xi_0\right)$ is a $\mathbb{K}_{q-1}\left(\xi_1\right)$-vector space *i.e.* there exists a vector subspace $\mathcal{W}$ of $E \otimes_{\mathbb{K}} \mathbb{K}_{q-1}$ such that $\mathcal{W} \otimes_{\mathbb{K}_{q-1}\left(\xi_1\right)} \mathbb{K}_q\left(\xi_0\right) = \mathcal{F}_0\left(\xi_0\right)$. Thus there exists a torsion-free coherent subsheaf $\mathcal{F}_1$ of $\left(h_{q-1} \circ \ldots \circ h_0\right)^* E$ such that $h_q^*\mathcal{F}_1 = \mathcal{F}_0$ (Lemma 2.2.2). Iterating this reasoning $(q-1)$-times, we determine a torsion-free coherent subsheaf $\mathcal{F}_q$ of h_0^*E such that $\left(h_q \circ \ldots \circ h_1\right)^* \mathcal{F}_q = \mathcal{F}_0$. Let G_0 be the Galois group of $\mathbb{K}_0$ over $\mathbb{K}$, again each $\sigma_0 \in G_0$ induces an automorphism $\widetilde{\sigma}_0$ of X_0 over X such $\widetilde{\sigma}_0^*\mathcal{F}_q = \mathcal{F}_q$. By [24, Theorem 9.28] there exists a subsheaf $\widetilde{\mathcal{F}}$ of $\mathfrak{E}$ such that $h_0^*\widetilde{\mathcal{F}} = \mathcal{F}_q$, hence $g_2^*\widetilde{\mathcal{F}} = \mathcal{F}_0$. By Lemma 2.2.3 $\widetilde{\mathcal{F}}$ is a torsion-free Higgs subsheaf of $\mathfrak{E}$ such that

$$\mu\left(\widetilde{\mathcal{F}}\right) = \mu\left(\mathcal{F}_0\right) > \mu\left(g_2^*\mathfrak{E}\right) = \mu(\mathfrak{E})$$

so that $\mathfrak{E}$ is an unstable Higgs sheaf. Q.e.d.

Theorem 2.2.5. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be torsion-free Higgs sheaves over (X, H). If both are H-semistable then $\left(\mathfrak{E}_1 \otimes \mathfrak{E}_2\right)/torsion$ is H-semistable too.*

Proof. Repeating the proof of Lemma 2.2.1, there exist an algebraically closed subfield $\mathbb{K}_0$ of $\mathbb{K}$, a variety $X_{\mathbb{K}_0}$ over $\mathbb{K}_0$, a line bundle H_0 over $X_{\mathbb{K}_0}$ and Higgs bundles $\mathfrak{F}_{0,h}$ over $X_{\mathbb{K}_0}$, such that the following diagram is Cartesian

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d] & X_{\mathbb{K}_0} \arrow[d] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}\left(\mathbb{K}_0\right) \end{tikzcd} ,$$

$f_0^*H_0 = H$ and $\mathfrak{E}_{0,h} = f_0^*\mathfrak{F}_{0,h}$, where $h \in \{1, 2\}$. Moreover, by Proposition 2.1.5 and [1, tag 0D2P] H_0 is a polarization of $X_{\mathbb{K}_0}$. Since $\mathbb{K}_0$ is a subfield of $\mathbb{C}$, up to isomorphism, we change the base of $X_{\mathbb{K}_0}$ and have the following Cartesian diagram

$$\begin{tikzcd} X_{\mathbb{C}} = X_{\mathbb{K}_0} \times_{\mathrm{Spec}(\mathbb{K}_0)} \mathrm{Spec}(\mathbb{C}) \arrow[r, "f"] \arrow[d] & X_{\mathbb{K}_0} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}\left(\mathbb{K}_0\right) \end{tikzcd} .$$

By the previous lemma, $f^*\mathfrak{E}_{0,1}$ and $f^*\mathfrak{E}_{0,2}$ are f^*H_0-semistable Higgs bundles over $X_{\mathbb{C}}$ hence $\left(f^*\mathfrak{E}_{0,1} \otimes f^*\mathfrak{E}_{0,2}\right)/torsion = f^*\left(\left(\mathfrak{E}_{0,1} \otimes \mathfrak{E}_{0,2}\right)/torsion\right)$ is f^*H_0-semistable as well ([5, Proposition 4.5] or [34, Theorem 5.4] equivalently). Again, by the previous lemma $\left(\mathfrak{E}_{0,1} \otimes \mathfrak{E}_{0,2}\right)/torsion$ is H_0-semistable hence $\left(\mathfrak{E}_1 \otimes \mathfrak{E}_2\right)/torsion = f^*\left(\left(\mathfrak{E}_{0,1} \otimes \mathfrak{E}_{0,2}\right)/torsion\right)$ is H-semistable too. Q.e.d.

2.3 Properties of Harder-Narasimhan filtration of Higgs sheaves, Part II

Applying the previous theorem we can prove other two results on semistable torsion-free Higgs sheaves, thus the title of this section.

The first lemma uses the notion of maximal destabilizing Higgs subsheaf and it relates the semistability of torsion-free Higgs sheaves to particular Higgs sheaves.

Lemma 2.3.1. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf on X. The* Higgs sheaf $\mathrm{End}(\mathfrak{E}) = (\mathfrak{E} \otimes \mathfrak{E}^\vee)$ */torsion is semistable if and only if $\mathfrak{E}$ is semistable.*

Proof. Let $\mathfrak{E}$ be semistable. By Lemma 1.2.9 and by Theorem 2.2.5 $\mathrm{End}(\mathfrak{E})$ is semistable. *Vice versa*, let $\mathrm{End}(\mathfrak{E})$ be semistable and let $\mathfrak{E}$ be unstable. By Lemma 1.2.9 also $\mathfrak{E}^\vee$ is unstable. Let $\mathfrak{E}_1$ and $\mathfrak{E}_{1^\vee}$ be the maximal destabilizing Higgs subsheaves of $\mathfrak{E}$ and $\mathfrak{E}^\vee$, respectively. By hypothesis

$$\begin{gathered}
\mu(\mathfrak{E}_1) > \mu(\mathfrak{E}), \mu\left(\mathfrak{E}_{1^\vee}\right) > \mu\left(\mathfrak{E}^\vee\right), \\
\mu(\mathfrak{E}_1 \otimes \mathfrak{E}_{1^\vee}) = \frac{\operatorname{rank}(\mathfrak{E}_{1^\vee}) \deg(\mathfrak{E}_1) + \operatorname{rank}(\mathfrak{E}_1) \deg(\mathfrak{E}_{1^\vee})}{\operatorname{rank}(\mathfrak{E}_1) \operatorname{rank}(\mathfrak{E}_{1^\vee})} = \mu(\mathfrak{E}_1) + \mu(\mathfrak{E}_{1^\vee}) > \\
> \mu(\mathfrak{E}) + \mu\left(\mathfrak{E}^\vee\right) = \mu(\mathrm{End}(\mathfrak{E})) = 0
\end{gathered}$$

i.e. $\mathfrak{E}_1 \otimes \mathfrak{E}_{1^\vee}$ is a Higgs subsheaf which destabilizes $\mathrm{End}(\mathfrak{E})$ in contradiction with the assumption. From all this we have the claim. Q.e.d.

The next proposition complete the list given by Proposition 1.4.6.

Proposition 2.3.2. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be torsion-free Higgs sheaves over X.* $\mu_{\min}(\mathfrak{E}_1 \otimes \mathfrak{E}_2) = \mu_{\min}(\mathfrak{E}_1) + \mu_{\min}(\mathfrak{E}_2)$, $\mu_{\max}(\mathfrak{E}_1 \otimes \mathfrak{E}_2) = \mu_{\max}(\mathfrak{E}_1) + \mu_{\max}(\mathfrak{E}_2)$.

Proof. Since:

$$\mu(\mathfrak{E}_{1,1} \otimes \mathfrak{E}_{2,1}) = \frac{\operatorname{rank}(\mathfrak{E}_{2,1}) \deg(\mathfrak{E}_{1,1}) + \operatorname{rank}(\mathfrak{E}_{1,1}) \deg(\mathfrak{E}_{2,1})}{\operatorname{rank}(\mathfrak{E}_{1,1}) \operatorname{rank}(\mathfrak{E}_{2,1})} = \mu(\mathfrak{E}_{1,1}) + \mu(\mathfrak{E}_{2,1})$$

by the proof of Theorem 1.4.2, $\mu_{\max}(\mathfrak{E}_1 \otimes \mathfrak{E}_2) \geq \mu_{\max}(\mathfrak{E}_1) + \mu_{\max}(\mathfrak{E}_2)$. On the other hand, by Theorem 2.2.5 $\mathfrak{E}_{1,1} \otimes \mathfrak{E}_{2,1}$ is a semistable Higgs subsheaf of $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ and therefore $\mu_{\max}(\mathfrak{E}_1 \otimes \mathfrak{E}_2) = \mu_{\max}(\mathfrak{E}_1) + \mu_{\max}(\mathfrak{E}_2)$. Analogously, we prove the other equality. Q.e.d.

Chapter 3

On H-ample, H-nef and H-nflat Higgs bundles

The contents of this chapter are mainly based on the papers [9] and [10] written in collaboration with Ugo Bruzzo and Beatriz Graña Otero, unless otherwise indicated.

3.1 Higgs-Grassmann schemes

Let $\mathfrak{E} = (E, \varphi)$ be a rank $r \geq 2$ Higgs bundle over a smooth irreducible projective variety X of dimension n, and let $s \in \{1, \dots, r-1\}$ be an integer number. Let $p_s \colon \mathrm{Gr}_s(E) \to X$ be the *Grassmann bundle* parametrizing rank s locally free quotients of E (see [25]). Consider the short exact sequence of vector bundles over $\mathrm{Gr}_s(E)$

$$0 \longrightarrow S_{r-s,E} \xrightarrow{\eta} p_s^* E \xrightarrow{\epsilon} Q_{s,E} \longrightarrow 0\,, \tag{3.1}$$

where $S_{r-s,E}$ is the *universal rank $r-s$ subbundle* and $Q_{s,E}$ is the *universal rank s quotient bundle of* $p_s^* E$, respectively.

These Grassmann bundles enjoy the following universal property.

Theorem 3.1.1 (Universal Property of Grassmann Bundle). *Let Y be a scheme over* $\mathrm{Spec}(\mathbb{K})$*, let $f \colon Y \to X$ be a morphism and let Q be a rank s quotient bundle of $f^* E$. Then there exists a unique morphism $g \colon Y \to \mathrm{Gr}_s(E)$ such that $Q = g^* Q_{s,E}$ and $f = p_s \circ g$.*

For $s = 1$ theorem is [30, Proposition II.7.12] and for any s is [40, Proposition 1.2].

With the aim to extend these properties to Higgs bundles, Bruzzo and Hernández Ruipérez have introduced in [15] closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subseteq \mathrm{Gr}_s(E)$ (the *s-th Higgs-Grassmann* *schemes of* $\mathfrak{E}$) which parametrize the rank s Higgs quotient bundles of $\mathfrak{E}$. These schemes are defined as the zero loci of the composite morphisms

$$(\epsilon \otimes \mathrm{Id}) \circ \psi \circ \eta \colon S_{r-s,E} \to Q_{s,E} \otimes p_s^*\Omega_X^1$$

where $\psi\colon p_s^*E \to p_s^*E \otimes p_s^*\Omega_X^1$ is the pullback of the morphism φ via p_s.

Theorem 3.1.2. *The restriction of the sequence* (3.1) *to* $\mathfrak{Gr}_s(\mathfrak{E})$ *yields a universal short exact sequence*

$$0 \longrightarrow \mathfrak{S}_{r-s,\mathfrak{E}} \xrightarrow{\eta} \rho_s^*\mathfrak{E} \xrightarrow{\epsilon} \mathfrak{Q}_{s,\mathfrak{E}} \longrightarrow 0,$$

where $\mathfrak{Q}_{s,\mathfrak{E}} = Q_{s,E|\,\mathfrak{Gr}_s(\mathfrak{E})}$ *is equipped with the quotient Higgs field induced by* $\rho_s^*\varphi$ *(here* $\rho_s = p_{s|\,\mathfrak{Gr}_s(\mathfrak{E})}\colon \mathfrak{Gr}_s(\mathfrak{E}) \to X$*).*

Proof. Let us consider

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_{r-s,E} & \xrightarrow{\eta} & p_s^*E & \xrightarrow{\epsilon} & Q_{s,E} & \longrightarrow & 0\,. \\
 & & & & \downarrow{\scriptstyle\psi} & & & & \\
0 & \longrightarrow & S_{r-s,E} \otimes p_s^*\Omega_X^1 & \xrightarrow[\eta\otimes\mathrm{Id}]{} & p_s^*E \otimes p_s^*\Omega_X^1 & \xrightarrow[\epsilon\otimes\mathrm{Id}]{} & Q_{s,E} \otimes p_s^*\Omega_X^1 & \longrightarrow & 0
\end{array}$$

By construction $\ker(\epsilon \otimes \mathrm{Id}) = S_{r-s,E} \otimes p_s^*\Omega_X^1$, then by the couniversal property of the kernel there exists a unique morphism $\psi_0 : S_{r-s,E|\,\mathfrak{Gr}_s(\mathfrak{E})} \to (S_{r-s,E} \otimes p_s^*\Omega_X^1)_{|\,\mathfrak{Gr}_s(\mathfrak{E})}$ of $\mathcal{O}_{\mathfrak{Gr}_s(\mathfrak{E})}$-modules which makes the diagram commutative. So that $S_{r-s,E|\,\mathfrak{Gr}_s(\mathfrak{E})} = \mathfrak{S}_{r-s,\mathfrak{E}}$ is a Higgs subbundle of $\rho_s^*\mathfrak{E}$ and therefore $\mathfrak{Q}_{s,\mathfrak{E}}$ is the corresponding Higgs quotient bundle. Q.e.d.

Also the Higgs-Grassmann schemes enjoy a universal property similar to that of the Grassmann bundles.

Theorem 3.1.3 (**Universal Property of Higgs-Grassmann schemes**)**.** *Let* Y *be a scheme, let* $f\colon Y \to X$ *be a morphism and let* $\mathfrak{Q}$ *be a rank* s *Higgs quotient bundle of* $f^*\mathfrak{E}$*. Then there exists a unique morphism* $g\colon Y \to \mathfrak{Gr}_s(E)$ *such that* $\mathfrak{Q} = g^*\mathfrak{Q}_{s,\mathfrak{E}}$ *and* $f = p_s \circ g$*.*

Proof. Let $\mathfrak{Q} = (Q, \widetilde{\varphi})$ be a rank s Higgs quotient bundle of $f^*\mathfrak{E}$; by the universal property of the Grassmann bundles, there exists a unique morphism $g\colon Y \to \mathrm{Gr}_s(E)$ such that $g^*Q_{s,E} = Q$ and $f = p_s \circ g$. Thus

$$\begin{array}{ccccccc}
g^*S_{r-s,E} & \xrightarrow{g^*\eta} & f^*E & \xrightarrow{g^*\epsilon} & g^*Q_{s,E} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle g^*\psi} & & \downarrow & & \\
g^*S_{r-s,E} \otimes f^*\Omega_X^1 & \xrightarrow[g^*\eta\otimes\mathrm{Id}]{} & f^*E \otimes f^*\Omega_X^1 & \xrightarrow[g^*\epsilon\otimes\mathrm{Id}]{} & g^*Q_{s,E} \otimes f^*\Omega_X^1 & \longrightarrow & 0
\end{array}$$

is a commutative diagram, hence $g^*\left((\epsilon \otimes \mathrm{Id}) \circ \psi \circ \eta\right) = 0$. This is possible if and only if g takes values in $\mathfrak{Gr}_s(\mathfrak{E})$, or equivalently if and only if $Q = g^*Q_{s,E|\,\mathfrak{Gr}_s(\mathfrak{E})}$, which is the claim. Q.e.d.

3.1.1 On the first Higgs-Grassmann scheme

In principle, $\mathfrak{Gr}_s(\mathfrak{E})$ could be empty. On the other hand, if $\mathfrak{Gr}_s(\mathfrak{E})$ is not empty then it may be neither smooth, nor reduced, nor equidimensional, nor irreducible. In this thesis, we prove the existence of an irreducible component Z of $\mathfrak{Gr}_1(\mathfrak{E})$ which surjects onto X when $\operatorname{rank}(\mathfrak{E}) \in \{2,3\}$. This last condition is satisfied at least in the following cases:

- $\operatorname{rank}(E) = 2$ and $\mathbb{K} = \mathbb{C}$, by [14, Corollary 4.3];
- $\mathfrak{E}$ has a Higgs quotient line bundle, *i.e.* $\rho_1 \colon \mathfrak{Gr}_1(\mathfrak{E}) \to X$ has a section.

In the following, we propose another proof of [14, Corollary 4.3] which works furthermore on any algebraically closed field of characteristic 0.

Proposition 3.1.4. *Let $\mathfrak{E}$ be a rank 2 Higgs bundle. $\mathfrak{Gr}_1(\mathfrak{E})$ contains a projective variety which surjects onto X.*

Proof. By [15, Subsection 3.2], $\mathfrak{Gr}_1(\mathfrak{E})$ is pointwise the intersection of $n\binom{2}{2} = n$ hyperquadrics in $\mathrm{Gr}_1(E)$; hence if $n = 1$ then $\mathfrak{Gr}_1(\mathfrak{E})$ contains an irreducible component of dimension at least 1 which surjects onto X.

Let $n \geq 2$ and let L be an ample line bundle over X. There exists $m \geq 1$ such that a basis of $\mathrm{H}^0(X, mL)$ defines a closed embeddings of X in $\mathbb{P}^N$ for some $N \geq 1$. Let $x \in X$ be a closed point. By Bertini's Theorem (cfr. [30, Corollary III.10.9 and Exercise III.11.3]), there exist ample smooth divisors $D_1, \ldots, D_{n-1} \in |mL|$ such that $C = D_1 \cap \ldots \cap D_{n-1}$ is a smooth projective curve in X and $x \in C$. By the previous step, we find an irreducible component Y of $\mathfrak{Gr}_1\left(\mathfrak{E}_{|C}\right)$ of dimension at least 1. Let us consider the resolution $\left(\widetilde{Y}_{red}, \beta\right)$ of Y_{red}[1] (see [32, Main Theorem I]), it contains a smooth projective curve. Consider the following commutative diagram

$$\begin{tikzcd}
\widetilde{Y}_{red} \arrow[r, "\beta"] \arrow[drrr, "f"'] & Y_{red} \arrow[r, hook] & Y \arrow[r, hook] & \mathfrak{Gr}_1\left(\mathfrak{E}_{|C}\right) \arrow[r, hook] \arrow[d, "\rho_{1|C}"'] & \mathfrak{Gr}_1(\mathfrak{E}) \arrow[d, "\rho_1"] \\
 & & & C \arrow[r, hook] & X,
\end{tikzcd}$$

by construction $f^*\mathfrak{E}$ has a locally free rank 1 Higgs quotient sheaf. By the universal property of $\mathfrak{Gr}_1(\mathfrak{E})$, there exists a unique morphism $\psi_f \colon \widetilde{Y}_{red} \to \mathfrak{Gr}_1(\mathfrak{E})$ such that $f = \rho_1 \circ \psi_f$, where $\rho_1 \colon \mathfrak{Gr}_1(\mathfrak{E}) \to X$ has been defined above. Since $f\left(\widetilde{Y}_{red}\right) = C$ then $\mathfrak{Gr}_1(\mathfrak{E})$ contains at least an irreducible curve which surjects onto C. In other words, for any closed point

[1] Since Y_{red} is a reduced irreducible closed subscheme of a projective scheme, it is a projective variety.

$x \in X$, $\mathfrak{Gr}_1(\mathfrak{E})$ contains a curve which image via ρ_1 contains x. Thus $\mathfrak{Gr}_1(\mathfrak{E})$ surjects onto X, this is possible if and only if $\mathfrak{Gr}_1(\mathfrak{E})$ has an irreducible component Z of dimension at least n; in particular Z_{red} is a projective variety which surjects onto X. Q.e.d.

Following the same idea, we prove the next proposition which is analogous to the previous one and it works for the rank 3 case.

Proposition 3.1.5. *Let $\mathfrak{E}$ be a rank 3 Higgs bundle. Then $\mathfrak{Gr}_1(\mathfrak{E})$ contains a projective variety Z which surjects onto X.*

Proof. By [15, Subsection 3.2], $\mathfrak{Gr}_1(\mathfrak{E})$ is locally the intersection of $n\binom{3}{2} = 3n$ hyperquadrics in $\mathrm{Gr}_1(E)$. Analysing the ideal of this intersection in *Macaulay2*, one proves that this has coheight at least 1; *i.e.* $\mathfrak{Gr}_1(\mathfrak{E})$ has a 1-dimensional irreducible component Z which surjects onto C, and Z_{red} a projective subvariety.

Let $n \geq 2$. One repeats the same reasoning of the previous proposition and concludes. Q.e.d.

3.2 H-ample Higgs bundles

We start this section recalling the definition of *H-ample Higgs bundle* given in [11]. We require the ampleness of the determinant line bundle and recursively the H-ampleness of all universal Higgs quotient bundles.

Definition 3.2.1 (see [11, Definition 2.3]). A Higgs bundle $\mathfrak{E} = (E, \varphi)$ of rank one is said to be *Higgs-ample* (*H-ample*, for short) if E is ample in the usual sense. If $\mathrm{rank}(\mathfrak{E}) \geq 2$, we inductively define H-ampleness by requiring that

a) all Higgs bundles $\mathfrak{Q}_{s,\mathfrak{E}}$ are H-ample for all s, and

b) the determinant line bundle $\det(E)$ is ample.

The condition on the determinant cannot be omitted. In order to prove this statement via an example we start by proving the following proposition.

Proposition 3.2.2 (cfr. [15, Proposition 3.7]). *Let $\mathfrak{E} = (E = L_1 \oplus L_2, \varphi)$ be a rank* 2 nilpotent[2] *Higgs bundle over (X, H) where H is a polarization of X, such that $\varphi(L_1) \subseteq L_2 \otimes \Omega^1_X$ and $\varphi(L_2) = 0$. Then*

[2] A Higgs bundle is *nilpotent* if there is a decomposition $E = \bigoplus_{i=1}^{m} E_i$ as direct sum of subbundles such that $\varphi(E_i) \subseteq E_{i+1} \otimes \Omega^1_X$ for $i \in \{1, \ldots, m-1\}$ and $\varphi(E_m) = 0$ (see [15]).

a) $\varphi = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \neq 0$ *only if* $\deg(L_1) \leq \deg(L_2) + \mu(\Omega^1_X)$, *where* $c \in \mathrm{Hom}_{\mathcal{O}_X}(L_1, L_2 \otimes \Omega^1_X)$ *is a monomorphism;*

b) $\mathfrak{Gr}_1(\mathfrak{E})$ *coincides with* $\mathrm{Gr}_1(L_1)$.

Proof. For our aims we assume $\varphi \neq 0$. Let

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$\begin{aligned} a &\in \mathrm{Hom}_{\mathcal{O}_X}\left(L_1, L_1 \otimes \Omega^1_X\right), b \in \mathrm{Hom}_{\mathcal{O}_X}\left(L_2, L_1 \otimes \Omega^1_X\right), \\ c &\in \mathrm{Hom}_{\mathcal{O}_X}\left(L_1, L_2 \otimes \Omega^1_X\right), d \in \mathrm{Hom}_{\mathcal{O}_X}\left(L_2, L_2 \otimes \Omega^1_X\right); \end{aligned}$$

by hypothesis, one has $a = 0$, $b = 0$, $d = 0$. Trivially $\left(L_2, \varphi_{|L_2} = 0\right)$ is a Higgs subbundle of $\mathfrak{E}$ and therefore $\mathfrak{L}_1 = (L_1, \widetilde{\varphi})$ is a Higgs quotient bundle of $\mathfrak{E}$.

Let $\mathfrak{K} = \left(\mathcal{K}, \varphi_{|\mathcal{K}}\right)$ be a Higgs subsheaf of $\mathfrak{E}$ such that the corresponding Higgs quotient sheaf $\mathfrak{Q}_0 = (\mathcal{Q}_0, \widetilde{\varphi}_0)$ is locally free and has rank 1. Thus $\mathcal{K}$ is a locally free sheaf, because kernel of an epimorphism of locally free sheaves on a Noetherian scheme. By hypothesis $\mathfrak{K} = (\mathfrak{K} \cap L_1) \oplus (\mathfrak{K} \cap L_2) \equiv \mathfrak{K}_1 \oplus \mathfrak{K}_2$, hence $\varphi(\mathcal{K}_2) = 0$ and $\varphi(\mathcal{K}_1) \subseteq \mathcal{K}_2 \otimes \Omega^1_X$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are the underlying coherent sheaves to $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively.

One has two possibilities:

1) $\mathcal{K}_1(\xi) = 0$ *i.e.* $\mathcal{K}_1$ is a torsion sheaf, hence $\mathcal{K}_1 = \underline{0}_X$ because L_1 is torsion-free. It follows that $\mathrm{rank}(\mathcal{K}_2) = 2$ and repeating the reasoning of Lemma 2.2.2 one has $\mathcal{K}_2 = L_2$.

2) $\mathcal{K}_1(\xi) \neq 0$ *i.e.* $\mathcal{K}_1$ is torsion-free then $\mathcal{K}_1 = L_1$. In this case it has to be $\mathrm{rank}(\mathcal{K}_2) = 0$, hence by the same reasoning $\mathcal{K}_2 = \underline{0}_X$. This force φ to be zero and this is impossible because we are supposing the contrary.

From all this, $(L_1, 0)$ is the only Higgs quotient bundle of $\mathfrak{E}$ and the claim follows. Q.e.d.

Here we are in position to give an example of negative degree Higgs bundle whose first universal Higgs quotient bundle is ample.

Example 3.2.3 (cfr. [11, Example 2.5])**.** Let X be a smooth projective curve of genus $g \geq 2$, and let $\mathfrak{E} = (E = L_1 \oplus L_2, \varphi)$ be a rank 2 nilpotent Higgs bundle over X such that $\varphi(L_1) \subseteq L_2 \otimes \Omega^1_X$ and $\varphi(L_2) = 0$. By the previous proposition, $\mathfrak{E}$ has only a Higgs quotient bundles which is $(L_1, 0)$. If we take $\deg(L_1) = 1$ and $\deg(L_2) = -2$, then $\mathfrak{E}$ is a Higgs bundle such that $\deg(E) = -1$ and $\mathfrak{Q}_{1,\mathfrak{E}}$ is an ample line bundle. △

Remark 3.2.4.

a) Definition 3.2.1 implies that $\forall k \in \{1,\dots,n\}$, $\int_X c_1(E)^k \cdot H^{n-k} > 0$, where H is a polarization of X. Moreover if $\mathfrak{E} = (E, \varphi)$, with E ample in the usual sense, then $\mathfrak{E}$ is H-ample. If $\varphi = 0$, the Higgs bundle $\mathfrak{E} = (E, 0)$ is H-ample if and only if E is ample in the usual sense.

b) The recursive condition definition of H-ampleness can be recast as follows.
Let $1 \leq s_1 < s_2 < \dots < s_k < r$ and let $\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}$ be the rank s_1 universal Higgs quotient bundle over $\mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\dots,s_k),\mathfrak{E}}\right)$, obtained by taking the successive universal Higgs quotient bundles of $\mathfrak{E}$ of rank s_k, then s_{k-1}, all the way to rank s_1. The H-ampleness condition for $\mathfrak{E}$ is equivalent to requiring that all line bundles $\det(\mathfrak{E})$ and $\det\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)$ are ample. $\diamondsuit$

We prove now some properties of H-ample Higgs bundles which will be useful in the sequel. These extend the properties given in [11] and the *Barton-Kleiman Criterion for Ampleness* ([45, Proposition 6.1.18.(ii)]) to the Higgs bundles framework.

Proposition 3.2.5.

a) Let $f\colon Y \to X$ be a finite morphism of smooth projective varieties. If $\mathfrak{E}$ is H-ample then $f^*\mathfrak{E}$ is H-ample. Moreover, if f is also surjective and $f^*\mathfrak{E}$ is H-ample then $\mathfrak{E}$ is H-ample.

b) Let $\mathfrak{E}$ be H-ample then every quotient Higgs bundle of $\mathfrak{E}$ is H-ample.

Proof. (a). (cfr. [11, Proof of Proposition 2.6.(ii)]) In the rank one case, we apply [45, Proposition 1.2.13 and Corollary 1.2.28]. In the higher rank case, we first note that the condition on the determinant is fulfilled because $f^*\det(E) = \det(f^*E)$. By the functoriality of Higgs-Grassmann schemes, f induces finite morphisms $\overline{f}_s\colon \mathfrak{Gr}_s(f^*\mathfrak{E}) \to \mathfrak{Gr}_s(\mathfrak{E})$ for all $s \in \{1,\dots,r-1\}$ such that $\mathfrak{Q}_{s,f^*\mathfrak{E}} = \overline{f}^*\mathfrak{Q}_{s,\mathfrak{E}}$. By induction on the rank of $\mathfrak{E}$, we conclude.

(b). Let $\mathfrak{Q}$ be a rank s Higgs quotient bundle of $\mathfrak{E}$; by the universal property of $\mathfrak{Gr}_s(\mathfrak{E})$ there exists a section $\sigma : X \to \mathfrak{Gr}_s(\mathfrak{E})$ such that $\mathfrak{Q} = \sigma^*\mathfrak{Q}_{s,\mathfrak{E}}$. Then $\sigma(X) = Y$ is a n-dimensional smooth projective subvariety of $\mathfrak{Gr}_s(\mathfrak{E})$; $\rho_{|Y}$ is an isomorphism with inverse σ, hence $\mathfrak{Q} = \sigma^*\mathfrak{Q}_{s,\mathfrak{E}}$ and by the previous part, $\mathfrak{Q}$ is H-ample. Q.e.d.

3.2.1 Some criteria for H-ampleness and some applications

Whenever we consider a morphism $f\colon C \to X$, we understand that C is an irreducible smooth projective curve.

As in the setting of ordinary vector bundles, the H-ampleness of a Higgs bundle can be tested by pulling-back to irreducible smooth projective curves.

Theorem 3.2.6. *Let $\mathfrak{E} = (E, \varphi)$ be a Higgs bundle on X. Fix an ample class $h \in \mathrm{N}^1(X)$. Then $\mathfrak{E}$ is H-ample if and only if*

a) the line bundle $\det(E)$ *is ample;*

b) there exists $\delta \in \mathbb{R}_{>0}$ such that for every finite morphism $f\colon C \to X$, the inequality

$$\mu_{\min}\left(f^*\mathfrak{E}\right) \geq \delta \int_C f^*h \tag{3.2}$$

holds, where $\mu_{\min}\left(f^*\mathfrak{E}\right)$ is defined via the relevant HN-filtration (1.3).

Proof. Let us assume that $\det(E)$ is ample and condition (3.2) holds. The H-ampleness of $\mathfrak{E}$ is equivalent to the ampleness of a collection of line bundles L_S, each on an iterated Higgs-Grassmann schemes (see Remark 3.2.4.b), that we denote generically by S, with projection $\pi_S\colon S \to X$. Let $q_S\colon \pi_S^*\mathfrak{E} \to L_S$ be the quotient morphism, let $g\colon C \to S$ be a finite morphism, and let $f = \pi_S \circ g$. We have a Higgs quotient $f^*\mathfrak{E} \to \mathfrak{Q}$, where $\mathfrak{Q} = g^*L_S$. By Proposition 1.4.6.b we have

$$\deg g^*L_S = \deg \mathfrak{Q} \geq \mu_{\min}(f^*E) \geq \delta \int_C f^*h,$$

so that by [45, Corollary 1.4.11] L_S is ample. As a consequence, $\mathfrak{E}$ is H-ample.

To prove the opposite implication, note that since $\mathfrak{E}$ is H-ample, the determinant $\det(E)$ is ample, and the class $c_1(E)$ is ample as well. Let $f\colon C \to X$ be a morphism. We have two cases, according to whether the Higgs bundle $f^*\mathfrak{E}$ is semistable or not. Let us start with the first case. By [45, Corollary 1.4.11], there exists $\epsilon \in \mathbb{R}_{>0}$ such that for every irreducible projective curve $\overline{C}$ in X we have

$$\int_{\overline{C}} c_1(E) \geq \epsilon \int_{\overline{C}} h.$$

Then

$$\mu_{\min}(f^*\mathfrak{E}) = \mu\left(f^*\mathfrak{E}\right) = \frac{1}{r}\int_C f^*c_1(E) = \frac{1}{r}\int_{\overline{C}} c_1(E) \geq \frac{\epsilon}{r}\int_{\overline{C}} h = \delta \int_C f^*h,$$

where $\overline{C}$ is the image of C in X, and $\delta = \dfrac{\epsilon}{r}$. In the second case, recall the HN-filtration of $f^*\mathfrak{E}$

$$0 \subsetneqq \mathfrak{E}_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}_{m-1} \subsetneqq \mathfrak{E}_m = f^*\mathfrak{E}.$$

By the universality of the Higgs-Grassmann schemes there is a lift $f_s\colon C \to \mathfrak{Gr}_s(\mathfrak{E})$ of f such that $\mathfrak{E}_m/\mathfrak{E}_{m-1} = f_s^*\mathfrak{Q}_{s,\mathfrak{E}}$, where $s = \operatorname{rank}(\mathfrak{E}_m/\mathfrak{E}_{m-1})$. Therefore,

$$\mu_{\min}(f^*\mathfrak{E}) = \frac{1}{s}\int_C f_s^*(c_1(\mathfrak{Q}_{s,\mathfrak{E}})).$$

Since $\mathfrak{Q}_{s,\mathfrak{E}}$ is H-ample, by [45, Corollary 1.4.11] there exists $\eta \in \mathbb{R}_{>0}$ such that the class

$$c_1(\mathfrak{Q}_{s,\mathfrak{E}}) - \eta\rho_s^*h$$

is ample for all possible values of s, so that

$$\frac{1}{s}\int_C f_s^*(c_1(\mathfrak{Q}_{s,\mathfrak{E}})) \geq \delta\int_C f^*h,$$

where $\delta = \dfrac{\eta}{r-1}$, thus proving the claim. Q.e.d.

Remark 3.2.7. To be clear, the "if part" of the previous theorem cannot be tested considering only the curves in X, *i.e.* considering only the closed embeddings $C \hookrightarrow X$. This holds because the ampleness of $\det\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)$'s (cfr. Remark 3.2.4.b) have to be tested on all curves in $\mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\ldots,s_k),\mathfrak{E}}\right)$. $\diamondsuit$

Corollary 3.2.8 (Barton-Kleiman-type criterion for H-ampleness). *Let $\mathfrak{E} = (E,\varphi)$ be a Higgs bundle on X. Fix an ample class $h \in \mathrm{N}^1(X)$. Then $\mathfrak{E}$ is H-ample if and only if*

a) the line bundle $\det(E)$ is ample;

b) there exists $\delta \in \mathbb{R}_{>0}$ such that for every finite morphism $f\colon C \to X$, and for every Higgs quotient bundle $\mathfrak{Q}$ of $f^*\mathfrak{E}$, the inequality

$$\deg(\mathfrak{Q}) \geq \delta\int_C f^*h$$

holds.

Proof. If $\mathfrak{E}$ is H-ample, by the previous theorem there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\deg(\mathfrak{Q}) = s\mu(\mathfrak{Q}) \geq s\mu_{\min}(f^*\mathfrak{E}) \geq s\delta\int_C f^*h \geq \delta\int_C f^*h$$

(here $s = \operatorname{rank}(\mathfrak{Q})$).

Conversely, let us assume that $\mathfrak{E}$ satisfies the conditions (a) and (b); let us call ϵ the constant. In particular, we can take for $\mathfrak{Q}$ the quotient $\mathfrak{E}_m/\mathfrak{E}_{m-1}$ of the HN-filtration of $f^*\mathfrak{E}$, so that

$$\mu_{\min}(f^*\mathfrak{E}) = \frac{\deg(\mathfrak{E}_m/\mathfrak{E}_{m-1})}{s} \geq \frac{\epsilon}{s}\int_C f^*h \geq \delta \int_C f^*h$$

where $\delta = \dfrac{\epsilon}{r}$. Therefore, $\mathfrak{E}$ is H-ample. Q.e.d.

Example 3.2.9. Let X be an irreducible smooth projective curve of genus g, and let $\mathfrak{E} = (L_1 \oplus L_2, \varphi)$ be the nilpotent Higgs bundle described in the Example 3.2.3. Let d_1 and d_2 be the degree of L_1 and L_2, respectively; we assume that $d_1 \leq 2g - 2 + d_2$. Let $d_1 + d_2 > 0$ so that $\det(E)$ is ample. Then $\mathfrak{E}$ is H-ample if and only if $\deg(L_1) = d_1 > 0$. Indeed, fix an ample divisor h on X, then

$$\deg(L_1) \geq \delta \deg(h) \tag{3.3}$$

choosing $0 < \delta \leq \dfrac{d_1}{\deg(h)}$. Now if $f \colon C \to X$ is a finite morphism, according to Proposition 3.2.2.b, $f^*\mathfrak{E}$ has an only one rank 1 Higgs quotient bundles which is $Q_1 = f^*L_1$ with zero Higgs field. Now, Inequality (3.3) implies

$$\deg(Q_1) \geq \delta \int_C f^*h$$

so that $\mathfrak{E}$ is H-ample whenever $d_1 > 0$. On the other hand, if $\mathfrak{E}$ is H-ample, then L_1 is ample by Proposition 3.2.5.b *i.e.* $d_1 > 0$.

If $g \geq 2$ one can arrange $d_1 > 0, d_2 \leq 0$ so that E is not ample as an ordinary bundle (for instance, with $g = 2$ we take $d_1 = 1$ and $d_2 = 0$). △

3.2.2 Applications of the H-ampleness criterion

Corollary 3.2.8 permits one to reduce the study of H-ample Higgs bundles to their finite pullbacks over irreducible smooth projective curves. This allows us to prove that the category of H-ample Higgs bundles is closed under extensions and tensor products.

Theorem 3.2.10. *Let $0 \to \mathfrak{E}_1 \to \mathfrak{E} \to \mathfrak{E}_2 \to 0$ be a short exact sequence of Higgs bundles over X with $\mathfrak{E}_1 = (E_1, \varphi_1)$ and $\mathfrak{E}_2 = (E_2, \varphi_2)$ H-ample. Then $\mathfrak{E}$ is H-ample.*

Proof. Since

$$\det(E) \cong \det(E_1) \otimes \det(E_2),$$

by hypothesis and [45, Corollary 6.1.16.(i)] it is an ample line bundle. Let $f\colon C \to X$ be a morphism and let $\mathfrak{Q}$ be a Higgs quotient bundle of $f^*\mathfrak{E}$. Let $f\colon C \to X$ be a finite morphism and let $\mathfrak{Q}$ be a quotient Higgs bundle of $f^*\mathfrak{E}$. We can form the following diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & f^*\mathfrak{E}_1 \arrow[r] \arrow[d] & f^*\mathfrak{E} \arrow[r] \arrow[d] & f^*\mathfrak{E}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{Q}_1 \arrow[r] \arrow[d] & \mathfrak{Q} \arrow[r] \arrow[d] & \mathfrak{Q}_2 \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Let $\overline{\mathfrak{Q}_2}$ be $\mathfrak{Q}_2$ modulo its torsion, and let $h \in \mathrm{N}^1(X)$ be an ample class. By Corollary 3.2.8 there exist $\delta_1, \delta_2 \in \mathbb{R}_{>0}$ such that

$$\deg\left(\mathfrak{Q}_1\right) \geq \delta_1 \int_C f^*h,\ \deg\left(\mathfrak{Q}_2\right) \geq \deg\left(\overline{\mathfrak{Q}_2}\right) \geq \delta_2 \int_C f^*h.$$

By letting $\delta = \delta_1 + \delta_2$ we have $\deg(\mathfrak{Q}) \geq \delta \displaystyle\int_C f^*h$. Again by Corollary 3.2.8, $\mathfrak{E}$ is H-ample. Q.e.d.

So we have proved also the following corollary.

Corollary 3.2.11. *Let $\mathfrak{E}_1 = (E_1, \varphi_1)$ and $\mathfrak{E}_2 = (E_2, \varphi_2)$ be H-ample Higgs bundles over X. Then $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$ is H-ample.*

Theorem 3.2.12. *Let $\mathfrak{E}_1 = (E_1, \varphi_1)$ and $\mathfrak{E}_2 = (E_2, \varphi_2)$ be H-ample Higgs bundles over X. Then $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ is H-ample.*

Proof. Fix an ample class $h \in \mathrm{N}^1(X)$. By Theorem 3.2.6 there exists $\delta_1, \delta_2 \in \mathbb{R}_{>0}$ such that for every finite morphism $f\colon C \to X$, the inequalities

$$\mu_{\min}\left(f^*\mathfrak{E}_1\right) \geq \delta_1 \int_C f^*h,\ \mu_{\min}\left(f^*\mathfrak{E}_2\right) \geq \delta_2 \int_C f^*h$$

hold. Letting $\delta = \delta_1 + \delta_2$, by Proposition 2.3.2 we obtain

$$\mu_{\min}\left(f^*(\mathfrak{E}_1 \otimes \mathfrak{E}_2)\right) = \mu_{\min}\left(f^*\mathfrak{E}_1\right) + \mu_{\min}\left(f^*\mathfrak{E}_2\right) \geq \delta_1 \int_C f^*h + \delta_2 \int_C f^*h = \delta \int_C f^*h.$$

Q.e.d.

By Theorem 3.2.12 and Proposition 3.2.5.b we can prove the following corollary.

Corollary 3.2.13. *Let $\mathfrak{E}$ be H-ample. Then for any $p \in \{1, \dots, r\}$ the p-th exterior power $\bigwedge^p \mathfrak{E}$ is H-ample, and for all positive integer numbers m the m-th symmetric power $S^m\mathfrak{E}$ is H-ample.*

Remark 3.2.14. More in general, applying a Schur functor S to an H-ample Higgs bundle $\mathfrak{E}$ one has again an H-ample Higgs bundle $S(\mathfrak{E})$. $\Diamond$

On the other hand, repeating the proof of [12, Proposition 4.8] one proves the following proposition.

Proposition 3.2.15. *Let $\mathfrak{E}$ be such that $S^m(\mathfrak{E})$ is H-ample for some $m \in \mathbb{N}_{\geq 2}$. Then $\mathfrak{E}$ is H-ample.*

3.3 H-nef and H-nflat Higgs bundles

We start this section recalling the definitions of *H-nef* and *H-nflat Higgs bundle* given in [13]. We require the nefness of the determinant line bundle and recursively the H-nefness of all universal Higgs quotient bundles.

Definition 3.3.1 (see [13, Definition A.2])**.** A Higgs bundle $\mathfrak{E} = (E, \varphi)$ of rank one is said to be *Higgs-numerically effective* (*H-nef*, for short) if E is *numerically effective* in the usual sense. If $\operatorname{rank}(\mathfrak{E}) \geq 2$, we inductively define H-nefness by requiring that

a) all Higgs bundles $\mathfrak{Q}_{s,\mathfrak{E}}$ are H-nef for all s, and

b) the determinant line bundle $\det(E)$ is nef.

$\mathfrak{E}$ is *Higgs-numerically* flat (*H-nflat*, for short) if $\mathfrak{E}$ and $\mathfrak{E}^\vee$ are both H-nef.

We recall that a line bundle L over a projective (non necessary smooth) variety X is *nef* if for any morphism $f \colon C \to X$ from an irreducible projective curve C the inequality $\displaystyle\int_C c_1(f^*L) \geq 0$ holds.

Remark 3.3.2.

a) The first Chern class of an H-nflat Higgs bundle is numerically zero, because the corresponding determinant bundle is nflat. Note that if E nef/nflat in the usual sense, then $\mathfrak{E}$ is H-nef/H-nflat. If $\varphi = 0$, the Higgs bundle $\mathfrak{E} = (E, 0)$ is H-nef/H-nflat if and only if E is nef/nflat in the usual sense.

b) The Higgs bundle described in Example 1.2.3 is semistable of degree 0 over a smooth projective curve, by [13, Lemma A.7] it is H-nflat. But it is not semistable as ordinary vector bundle and therefore it is not nflat. $\diamondsuit$

Even H-nef Higgs bundles satisfy properties analogous to those of H-ample Higgs bundles. These properties have been proved in [11, 12, 3]; here we list some of them for completeness.

Lemma 3.3.3. *Let $\mathfrak{E}$ be an H-nef Higgs bundle over X. The following statements hold.*

a) Let $f\colon Y \to X$ be a morphism of smooth projective varieties. Then $f^*\mathfrak{E}$ is H-nef ([11, Proposition 2.6.(ii)]). If f is also surjective and $f^*\mathfrak{E}$ is H-nef then $\mathfrak{E}$ is H-nef ([3, Lemma 3.4]).

b) Every quotient Higgs bundle of $\mathfrak{E}$ is H-nef ([3, Lemma 3.5]).

c) Tensor products of H-nef Higgs bundles are H-nef ([3, Theorem 3.6]).

d) Exterior and symmetric powers of H-nef Higgs bundles are H-nef (cfr. [12, Propositions 3.5, 4.4 and Lemma 4.5]).

e) $\mathfrak{E}$ is H-nef if and only if the Higgs bundle $\mathfrak{E}\otimes\mathcal{O}_X(D) = (E\otimes\mathcal{O}_X(D), \varphi\otimes Id)$ is H-ample for every ample Cartier $\mathbb{Q}$-divisor D in X ([11, Proposition 2.6.(i)]).

f) $\mathfrak{E}$ is H-nef if and only if for every finite morphism $f\colon C \to X$ one has $\mu_{\min}(f^*\mathfrak{E}) \geq 0$; where C is a smooth irreducible projective curve; $\mu_{\min}(f^*\mathfrak{E})$ is defined via the relevant HN-filtration (1.3) ([3, Lemma 3.3]).

g) Let $\mathbb{K} = \mathbb{C}$. Then the extensions of H-nef Higgs bundles are H-nef ([12, Propositions 3.9, 4.4 and Lemma 4.5]).

Remark 3.3.4. In [11], the first part of the statement a and the statement e are proved assuming that $\mathbb{K} = \mathbb{C}$, however these proofs work in general by [45, Propositions 6.1.2.(ii) and 6.1.8.(iv)]. $\diamondsuit$

Here we prove a corollary of Lemma 3.3.3.f, which will be used to extend Lemma 3.3.3.g to any algebraically closed field of characteristic 0.

Corollary 3.3.5. *$\mathfrak{E}$ is H-nef if and only if*

a) the line bundle $\det(E)$ is nef;

b) for every morphism $f\colon C \to X$, and every Higgs quotient $\mathfrak{Q}$ of $f^*\mathfrak{E}$, the inequality $\deg(\mathfrak{Q}) \geq 0$ holds.

Proof. Let $\mathfrak{E}$ be H-nef; by definition $\det(E)$ is nef. For any morphism $f\colon C \to X$, $f^*\mathfrak{E}$ is H-nef by Lemma 3.3.3.a, and its Higgs quotient bundles $\mathfrak{Q}$ are H-nef by Lemma 3.3.3.b hence $\deg(\mathfrak{Q}) \geq 0$.

Vice versa, let $\mathfrak{E}$ satisfy the hypotheses and let

$$\{0\} = \mathfrak{E}_0 \subsetneqq \mathfrak{E}_1 \subsetneqq \ldots \subsetneqq \mathfrak{E}_{m-1} \subsetneqq \mathfrak{E}_m = f^*\mathfrak{E}$$

be the HN-filtration of $f^*\mathfrak{E}$. By hypothesis $\mu_{\min}(f^*\mathfrak{E}) = \mu(\mathfrak{E}_m/\mathfrak{E}_{m-1}) \geq 0$, hence by Lemma 3.3.3.f we conclude. Q.e.d.

Example 3.3.6. Let X be an irreducible smooth projective curve of genus g, and let $\mathfrak{E} = (L_1 \oplus L_2, \varphi)$ be the nilpotent Higgs bundle described in the Example 3.2.3. Let d_1 and d_2 be the degree of L_1 and L_2, respectively; we assume that $d_1 \leq 2g - 2 + d_2$. Let $d_1 + d_2 \geq 0$ so that $\det(E)$ is nef. Then $\mathfrak{E}$ is H-nef if and only if $\deg(L_1) = d_1 \geq 0$. Indeed, if $f\colon C \to X$ is a finite morphism, according to Proposition 3.2.2.b, $f^*\mathfrak{E}$ has only one rank 1 Higgs quotient bundle which is $Q_1 = f^*L_1$ with zero Higgs field. Now

$$\deg(Q_1) \geq 0$$

so that $\mathfrak{E}$ is H-nef whenever $d_1 \geq 0$ by the previous corollary. On the other hand, if $\mathfrak{E}$ is H-nef, then L_1 is nef by Lemma 3.3.3.b *i.e.* $d_1 \geq 0$.

If $g \geq 2$ one can arrange $d_1 \geq 0, d_2 < 0$ so that E is not nef as an ordinary bundle (for instance, with $g = 2$ we take $d_1 = 1$ and $d_2 = -1$). △

Proposition 3.3.7. *Let* $0 \to \mathfrak{E}_1 \to \mathfrak{E} \to \mathfrak{E}_2 \to 0$ *be a short exact sequence of Higgs bundles over* X *with* $\mathfrak{E}_1 = (E_1, \varphi_1)$ *and* $\mathfrak{E}_2 = (E_2, \varphi_2)$ *H-nef. Then* $\mathfrak{E}$ *is H-nef.*

Proof. Repeating the proof of Theorem 3.2.10, we construct the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f^*\mathfrak{E}_1 \arrow[r, "i"] \arrow[d, "\pi_1"'] & f^*\mathfrak{E} \arrow[r, "p"] \arrow[d, "\pi"'] & f^*\mathfrak{E}_2 \arrow[r] \arrow[d, "\pi_2"'] & 0 \\
0 \arrow[r] & \mathfrak{Q}_1 \arrow[r, "\widetilde{i}"'] \arrow[d] & \mathfrak{Q} \arrow[r, "\widetilde{p}"'] \arrow[d] & \mathfrak{Q}_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where:

- $\mathfrak{Q}_1, \mathfrak{Q}, \mathfrak{Q}_2$ are Higgs quotient bundles of $f^*\mathfrak{E}_1, f^*\mathfrak{E}, f^*\mathfrak{E}_2$, respectively;
- the rows are exact.

Thus:

$$\deg(\mathfrak{Q}) = \deg(\mathfrak{Q}_1) + \deg(\mathfrak{Q}_2) \geq 0,$$

and by Corollary 3.3.5 we conclude. Q.e.d.

Category of H-nflat Higgs bundles satisfies other properties which have been proved in [13, 3, 44, 9].

Lemma 3.3.8.

a) If the pullback of $\mathfrak{E}$ via any $f\colon C \to X$ is semistable and $\int_C f^*c_1(E) = 0$, then $\mathfrak{E}$ is H-nflat (cfr. [13, Lemma A.7]).

b) Any H-nflat Higgs bundle is semistable ([13, Proposition A.8]).

c) Extensions of H-nflat Higgs bundles are H-nflat ([3, Proposition 3.1.(iii)]).

d) Tensor products of H-nflat Higgs bundles are H-nflat ([3, Proposition 3.1.(iv)]).

e) Kernels and cokernels of morphisms of H-nflat Higgs bundles are H-nflat Higgs bundles ([3, Propositions 3.7 and 3.8]).

f) Let $\mathfrak{E}$ be a Higgs bundle over X and let $\mathbb{K} = \mathbb{C}$. $\mathfrak{E}$ is H-nflat if and only if it is pseudostable *(*i.e.*, it has a filtration whose quotients are locally free and stable) and the quotients of the filtration are H-nflat ([9, Theorem 3.2]).*

Remark 3.3.9. The "if part" of Lemma 3.3.8.f follows by Lemma 3.3.8.c. Indeed, let $\mathfrak{E}$ be a Higgs bundle and let

$$0 = \mathfrak{F}_0 \subsetneqq \mathfrak{F} \subsetneqq \mathfrak{F}_1 \subsetneqq \dots \subsetneqq \mathfrak{F}_m \subsetneqq \mathfrak{F}_{m+1} = \mathfrak{E}$$

be a filtration of $\mathfrak{E}$ in Higgs subbundle whose quotients $\mathfrak{F}, \mathfrak{Q}_1, \dots, \mathfrak{Q}_{m+1}$ are locally free, stable and H-nflat. As explained in the proof of [9, Theorem 3.2], $\mathfrak{F}_1$ is H-nflat. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{F}_1 \longrightarrow \mathfrak{F}_2 \longrightarrow \mathfrak{Q}_2 \longrightarrow 0\,,$$

since $\mathfrak{F}_1$ and $\mathfrak{Q}_2$ are H-nflat, by Lemma 3.3.8.c $\mathfrak{F}_2$ is H-nflat. Iterating this reasoning, we prove that $\mathfrak{E}$ is H-nflat. $\Diamond$

Even if [13, Lemma A.7 and Proposition A.8] have been proved where $\mathbb{K} = \mathbb{C}$, we shall explain that this hypothesis is useless at page 46, in the sense this lemma works on algebraically closed field of characteristic 0. Thus, also the Lemmata 3.3.8.c and 3.3.8.d hold on any algebraically closed field of characteristic 0. The original proofs of the remaining lemmata continue to be valid in this extending setting, therefore we do not repeat them here. Moreover, we shall erase the hypothesis of $\mathbb{K} = \mathbb{C}$ in Lemma 3.3.8.f.

Chapter 4

Curve semistable Higgs bundles and positivity conditions

The contents of this chapter are mainly based on papers [9] and [10] written in collaboration with Ugo Bruzzo and Beatriz Graña Otero.

4.1 Curve semistable Higgs bundles

Let $\mathfrak{E} = (E, \varphi)$ be a rank $r \geq 2$ Higgs bundle over a smooth projective variety X of dimension n over an algebraically closed field of characteristic 0, whenever we consider a morphism $f\colon C \to X$, we understand that C is an irreducible smooth projective curve.

Definition 4.1.1. $\mathfrak{E}$ is *curve semistable* if for every morphism $f\colon C \to X$ the pullback Higgs bundle $f^*\mathfrak{E}$ is semistable.

The study of this class of Higgs bundle has started in [15] as a generalisation to Higgs bundles framework of Miyaoka's work [52, Section 3] on semistable vector bundles over smooth projective curves. There Bruzzo and Hernández Ruipérez have introduced the following numerical classes

$$\lambda_s(\mathfrak{E}) = \left[c_1\left(\mathcal{O}_{\mathrm{Gr}_1(Q_{s,\mathfrak{E}})}(1)\right)\right] - \frac{1}{r}\varpi_s^*(c_1(E)) \in N^1\left(\mathrm{Gr}_1\left(Q_{s,\mathfrak{E}}\right)\right) \tag{4.1}$$

$$\theta_s(\mathfrak{E}) = \left[c_1\left(Q_{s,\mathfrak{E}}\right)\right] - \frac{s}{r}\rho_s^*(c_1(E)) \in N^1\left(\mathfrak{Gr}_s(\mathfrak{E})\right) \tag{4.2}$$

where $s \in \{1, \ldots, r-1\}$, $\rho_s\colon \mathfrak{Gr}_s(\mathfrak{E}) \to X$ has been defined in the previous chapter and $\varpi_s\colon \mathrm{Gr}_1\left(Q_{s,\mathfrak{E}}\right) \to \mathfrak{Gr}_s(\mathfrak{E}) \xrightarrow{\rho_s} X$. Bruzzo, Graña Otero and Hernández Ruiperéz have proved in [15, 11, 13] the following theorems to which the following notions are premised.

We recall that a class $\gamma \in N^1(X)$ is *numerically effective* (*nef*, for short) if for any irreducible projective curve $C \subseteq X$ the inequality $\gamma \cdot [C] \geq 0$ holds. And we call *positive* a class $\gamma \in N^1(X)$ if for any irreducible projective curve $C \subseteq X$ the inequality $\gamma \cdot [C] > 0$ holds.

Theorem 4.1.2 (see [15, Theorem 1.2])**.** *$\mathfrak{E}$ is curve semistable if and only the classes $\theta_s(\mathfrak{E})$ are* nef *for any $s \in \{1, \dots, r-1\}$.*

Theorem 4.1.3 (see [15, Theorem 1.2])**.** *$\mathfrak{E}$ is curve semistable if and only if the classes $\lambda_s(\mathfrak{E})$ are* nef *for any $s \in \{1, \dots, r-1\}$.*

Remark 4.1.4. By definition $\lambda_1(\mathfrak{E}) = \theta_1(\mathfrak{E})$. Thus if $r = 2$ the previous theorems are the same. $\Diamond$

Before proving these theorems, we recall the following lemma.

Lemma 4.1.5 (see [15, Lemma 3.3])**.** *Let $f\colon Y \to X$ be a finite surjective morphism of smooth projective curves. Then $\mathfrak{E}$ is semistable if and only if $f^*\mathfrak{E}$ is semistable.*

Proof. If $\mathfrak{E}$ is unstable then there exists a torsion-free Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ such that $\mu(\mathfrak{F}) > \mu(\mathfrak{E})$, hence $\mu\left(f^*\mathfrak{F}\right) > \mu\left(f^*\mathfrak{E}\right)$ *i.e.* $f^*\mathfrak{E}$ is unstable. Let us assume $f^*\mathfrak{E}$ unstable, without loss of generality we can assume that f is a Galois covering[1] with Galois group G, *i.e.* the field extension $f^\#\colon \mathbb{K}(X) \to \mathbb{K}(Y)$ is normal (and separable) and the relevant Galois group is G. Let $\mathfrak{F} = \left(F, f^*\varphi_{|F}\right)$ be the maximal destabilizing Higgs subsheaf of $f^*\mathfrak{E}$. For any $g \in G$, $g^*\mathfrak{F}$ is a destabilizing Higgs subsheaf of $f^*\mathfrak{E}$ of maximal rank; however by the unicity of the HN-filtration of $f^*\mathfrak{E}$ it has to be $g^*\mathfrak{F} = \mathfrak{F}$. From all this, it follows that $F = f^*E_0$ for some destabilizing subbundle E_0 of E. By [30, Exercise III.9.3.a] f is a flat morphism, and since it is also surjective then f is (by definition) faithfully flat. Thus the composition $E_0 \otimes \Omega^1_X \to E \otimes \Omega^1_X \to (E/E_0) \otimes \Omega^1_X$ vanishes if and only if the composition $F \otimes f^*\Omega^1_X \to f^*E \otimes f^*\Omega^1_X \to (f^*E/F) \otimes f^*\Omega^1_X$ vanishes. Consider the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & F \arrow[r] \arrow[dr, "f^*\varphi"'] & f^*E \otimes f^*\Omega^1_X \arrow[r] \arrow[d] & (f^*E/F) \otimes f^*\Omega^1_X \arrow[d] & \\
0 \arrow[r] & F \otimes \Omega^1_Y \arrow[r] & f^*E \otimes \Omega^1_Y \arrow[r] & (f^*E/F) \otimes \Omega^1_Y \arrow[r] & 0
\end{tikzcd}$$

since $f^*\varphi_{|F}$ takes values in $F \otimes \Omega^1_Y$ we have the claim, *i.e.* $\mathfrak{E}$ is unstable. Q.e.d.

[1]In general, if $char(\mathbb{K}) > 0$ then we need also that f is separable as stated in [52, 15].

Proof of Theorem 4.1.2. Let $C \subseteq X$ be an irreducible projective curve and assume that the restriction $\theta_s(\mathfrak{E})_{|C}$ of $\theta_s(\mathfrak{E})$ to $\mathfrak{Gr}_s\left(\mathfrak{E}_{|C}\right)$ is nef. If $\mathfrak{F} = (F, \psi)$ is a rank s torsion-free Higgs quotient sheaf of $\mathfrak{E}_{|C}$ then by [57, Corollary at page 75] F is locally free. By Theorem 3.1.3 there exists a unique section $\sigma\colon C \to \mathfrak{Gr}_s\left(\mathfrak{E}_{|C}\right)$ such that $\mathfrak{F} = \sigma^*\mathfrak{Q}_{s,\mathfrak{E}|C}$, where $\mathfrak{Q}_{s,\mathfrak{E}|C}$ is the restriction of $\mathfrak{Q}_{s,\mathfrak{E}}$ to $\mathfrak{Gr}_s\left(\mathfrak{E}_{|C}\right)$. Then

$$0 \leq \theta_s(\mathfrak{E})_{|C} \cdot [\sigma(C)] = \deg(F) - \frac{s}{r}\deg\left(E_{|C}\right) = s\left(\mu(F) - \mu\left(E_{|C}\right)\right);$$

thus if any $\theta_s(\mathfrak{E})$ is nef then by Proposition 1.2.5 $\mathfrak{E}_{|C}$ is semistable, *i.e.* $\mathfrak{E}$ is curve semistable. *Vice versa*, let $\mathfrak{E}$ curve semistable and let us assume that $\theta_s(\mathfrak{E})$ is not nef for some $s \in \{1, \ldots, r-1\}$ *i.e.* there exists an irreducible projective curve $C' \subseteq \mathfrak{Gr}_s(\mathfrak{E})$ such that $\theta_s(\mathfrak{E}) \cdot [C'] < 0$. Under this hypothesis, C' is not contained in a fibre of ρ_s, hence it surjects onto a projective curve $C \subseteq X$. We may choose a projective curve C'' and a morphism $h\colon C'' \to C$ such that $\widetilde{C} = C'' \times_C C'$ is a union of projective curves C_j isomorphic to C (see the proof of [52, Theorem 3.1]); $\theta_s\left(h^*\mathfrak{E}\right) \cdot [C_j] < 0$ for any index j evidently. For clarity, we have the following Cartesian diagram

$$\begin{tikzcd}
\mathfrak{Gr}_s\left(h^*\mathfrak{E}\right) \arrow[r, "h_s"'] \arrow[d] & \mathfrak{Gr}_s\left(\mathfrak{E}_{|C}\right) \arrow[r, hook] \arrow[d] & \mathfrak{Gr}_s(\mathfrak{E}) \arrow[d, "\rho_s"] & C' \arrow[l, hook'] \\
C'' \arrow[r, "h"'] & C \arrow[r, hook, "i"'] & X &
\end{tikzcd},$$

by the universal property of fibre products $\widetilde{C} \hookrightarrow \mathfrak{Gr}_s\left(h^*\mathfrak{E}\right)$. Let $\mathfrak{E}_j = h_s^*\left(\rho_s^*\mathfrak{E}_{|C}\right)_{|C_j}$ and let $\mathfrak{Q}_j = \mathfrak{Q}_{s,h^*\mathfrak{E}|C_j}$. By Lemma 4.1.5 $\mathfrak{E}_j$ is a semistable Higgs bundle and in particular $\mu(\mathfrak{Q}_j) \geq \mu(\mathfrak{E}_j)$. On the other hand

$$0 > \theta_s\left(h^*\mathfrak{E}\right) \cdot [C_j] = \left(c_1(Q_j) - \frac{s}{r}\rho_s^*c_1(E)\right) \cdot [C_j] = s\left(\mu(Q_j) - \mu(E_j)\right)$$

where E_j and Q_j are the underlying vector bundles to $\mathfrak{E}_j$ and $\mathfrak{Q}_j$, respectively. But by Proposition 1.2.5 this is a contradiction with the assumptions, hence all classes $\theta_s(\mathfrak{E})$ have to be nef. Q.e.d.

Proof of Theorem 4.1.3. The class $\lambda_s(\mathfrak{E})$ may be regarded as the numerical class of the hyperplane bundle of the Higgs $\mathbb{Q}$-bundle $\mathfrak{F}_s = \mathfrak{Q}_{s,\mathfrak{E}} \otimes \rho_s^*\left(\det(E)^{-1/r}\right)$ over $\mathfrak{Gr}_s(\mathfrak{E})$. As consequence

$$c_1(\mathfrak{F}_s) = c_1\left(Q_{s,\mathfrak{E}}\right) - \frac{s}{r}\rho_s^*c_1(E) \Rightarrow \theta_s(\mathfrak{E}) = [c_1(\mathfrak{F}_s)] \in N^1\left(\mathfrak{Gr}_s(\mathfrak{E})\right).$$

From all this, the classes $\lambda_s(\mathfrak{E})$ are nef if and only if the classes $\theta_s(\mathfrak{E})$ are nef, *i.e.* if and only if $\mathfrak{E}$ is curve semistable (Theorem 4.1.2). Q.e.d.

Remark 4.1.6. Miming Definition 4.1.1, $\mathfrak{E}$ is *curve stable* if for every morphism $f\colon C \to X$ the pullback Higgs bundle $f^*\mathfrak{E}$ is stable.

By the previous proofs, the curve stability of $\mathfrak{E}$ implies the positivity of $\theta_s(\mathfrak{E})$ and $\lambda_s(\mathfrak{E})$ for any $s \in \{1, \dots, r-1\}$. $\diamondsuit$

Now we are position to prove that [13, Lemma A.7] works on any algebraically closed field of characteristic 0. This proof needs Theorem 4.1.2.

Proof of Lemma 3.3.8.a. Under these hypotheses, the classes $c_1\left(\mathcal{O}_{\mathrm{Gr}_1\left(Q_{s,\mathfrak{E}}\right)}(1)\right)$ are nef for any $s \in \{1, \dots, r-1\}$ (Theorem 4.1.3), hence their pullbacks to $\mathfrak{Gr}_s\left(f^*\mathfrak{Q}_{s,\mathfrak{E}}\right)$ are nef. In other words, using the notations introduced in Remark 3.2.4.b, $\mathfrak{Q}_{(1,s),\mathfrak{E}}$ is nef; and it remains only to prove that $\det\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)$ are nef for all strings of integer numbers $1 \le s_1 < s_2 < \ldots < s_k < r$. By construction, $\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}$ is a Higgs bundle over $\mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\cdots,s_k),\mathfrak{E}}\right)$ and there is a morphism $\rho_{(s_1,\cdots,s_k)}\colon \mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\cdots,s_k),\mathfrak{E}}\right) \to X$ such that $\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}$ is a rank s_1 Higgs quotient bundle of $\rho^*_{(s_1,\cdots,s_k)}\mathfrak{E}$. Thus there exists a unique morphism $g_{(s_1,\cdots,s_k)}\colon \mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\cdots,s_k),\mathfrak{E}}\right) \to \mathfrak{Gr}_{s_1}(\mathfrak{E})$ such that $\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}} = g^*_{(s_1,\cdots,s_k)}\mathfrak{Q}_{s_1,\mathfrak{E}}$ (Theorem 3.1.3). Therefore

$$\left[c_1\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)\right] = g^*_{(s_1,\cdots,s_k)}\left[c_1\left(\mathfrak{Q}_{s_1,\mathfrak{E}}\right)\right] = g^*_{(s_1,\cdots,s_k)}\theta_{s_1}(\mathfrak{E})$$

because $\int_C f^*c_1(E) = 0$, hence $\det\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)$ is nef because $\theta_{s_1}(\mathfrak{E})$ is nef by Theorem 4.1.2. In other words $\mathfrak{E}$ is H-nef. Repeating all this reasoning, considering that also $\mathfrak{E}^\vee$ is curve semistable by Lemma 1.2.9, $c_1\left(E^\vee\right) = -c_1(E)$ hence $\int_C f^*c_1\left(E^\vee\right) = 0$, we prove in the same way the H-nefness of $\mathfrak{E}^\vee$, *i.e.* $\mathfrak{E}$ is H-nflat. Q.e.d.

4.2 Curve semistable Higgs bundles whose discriminant class vanishes

We have introduced the curve semistable Higgs bundles because we would like to extend the following theorem to Higgs bundles.

Theorem 4.2.1 (cfr. [55, Theorem 2] and [15, Theorem 1.4])**.** *For a vector bundle E over a smooth projective variety X the following statements are equivalent:*

a) $\theta_1(E)$ is nef;

b) E is curve semistable;

c) E is semistable with respect to some polarization H and $c_2(\mathrm{End}(E)) = 0 \in \mathrm{A}^2(X)$;

d) E is semistable with respect to some polarization H and $\int_X c_2(\mathrm{End}(E)) \cdot H^{n-2} = 0$.

Proof. (a) is equivalent to (b). This is [52, Theorem 3.1] when $\dim X = 1$. Let $\dim X \geq 2$. If E is curve semistable then $\theta_1(E)$ is nef by Theorem 4.1.2. *Vice versa*, if $\theta_1(E)$ is nef, let $f\colon C \to X$ be a morphism. Consider the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Gr}_1(f^*E) \arrow[r, "\overline{f}"] \arrow[d, "p_{1|C} \equiv f^*(p_1)"'] & \mathrm{Gr}_1(E) \arrow[d, "p_1"] \\ C \arrow[r, "f"'] & X \end{tikzcd},$$

one has

$$\overline{f}^* \lambda_1(E) = \overline{f}^* \left(\left[c_1\left(Q_{1,E} \otimes p_1^*\left(\det(E)^{-1/r}\right)\right)\right]\right) = \left[c_1\left(Q_{1,f^*E} \otimes p_{1|C}^*\left(\det(E)^{-1/r}\right)\right)\right] = \lambda_1(f^*E).$$

Since $Q_{1,E} \otimes p_1^*\left(\det(E)^{-1/r}\right)$ is nef then its pullback $Q_{1,f^*E} \otimes p_{1|C}^*\left(\det(E)^{-1/r}\right)$ via f is nef (Lemma 3.3.3.a). Thus $\lambda_1(f^*E)$ is nef as well, and by [52, Theorem 3.1] f^*E is semistable. In other words, we have the claim.

(b) implies (c). By hypothesis, for any morphism $f\colon C \to X$, f^*E is semistable. By Lemma 2.3.1 $f^*\mathrm{End}(E) = f^*(E \otimes E^\vee) = f^*E \otimes f^*E^\vee$ is semistable *i.e.* $\mathrm{End}(E)$ is curve semistable. Since $c_1(\mathrm{End}(E)) = 0$ then $\mathrm{End}(E)$ is nflat (Lemma 3.3.8.a), hence it is semistable (Lemma 3.3.8.b) and this implies the semistability of E (Lemma 2.3.1). Finally [19, Propositions 1.2.9 and 1.3] proves that $c_2(\mathrm{End}(E)) = 0$.

(c) implies (b). If $\dim X = 1$ there is nothing to prove of course. Let $\dim X = n \geq 2$, repeating the reasoning of Theorem 2.2.5 we can assume $\mathbb{K} \subseteq \mathbb{C}$. Consider the following Cartesian diagram

$$\begin{tikzcd} X_{\mathbb{C}} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{K}) \end{tikzcd},$$

by hypothesis $\mathrm{End}(E)$ is a degree 0 semistable Higgs bundle with $c_2(\mathrm{End}(E)) = 0$, and by Lemma 2.2.4 $f^*\mathrm{End}(E)$ is a semistable vector bundle over $X_{\mathbb{C}}$ such that $c_2(f^*\mathrm{End}(E)) = 0$. If there exists a morphism $g\colon C \to X$ such that g^*E is unstable, then by Lemma 2.2.4 $\overline{g}^*E$ is unstable, where $\overline{g}$ is the base change morphism of g over $\overline{C} = C \times_{\mathbb{K}} \mathbb{C}$. This gives rise a contradiction: $\overline{g}^*\mathrm{End}(E)$ is the pullback of $f^*\mathrm{End}(E)$ over $\overline{C}$ and it is unstable, but by [15, Theorem 1.4] $\overline{g}^*\mathrm{End}(E)$ is semistable. To avoid this absurd, E is curve semistable.

(c) implies (d). This is trivial.

(d) implies (c). By Lemma 2.3.1 $\mathrm{End}(E)$ is semistable with $\int_X c_2(\mathrm{End}(E)) = 0$, while $c_1(\mathrm{End}(E)) = 0$ of course. Let

$$\{0\} = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_{m-1} \subsetneq F_m = \mathrm{End}(E),$$

be a JH-filtration of $\mathrm{End}(E)$, by [42, Corollary 6] this can be chosen in a way that the quotients $Q_i = F_i/F_{i-1}$ have vanishing Chern classes for any $i \in \{1, \ldots, m\}$. In other words, $\mathrm{End}(E)$ is an iterating extension of vector bundles with vanishing Chern classes, thus the same statement holds for the Chern classes of $\mathrm{End}(E)$; in particular $c_2(\mathrm{End}(E)) = 0$. Q.e.d.

Moreover, the previous theorem is equivalent to the following one.

Theorem 4.2.2 (cfr. [3, Corollary 3.2])**.** *On a smooth projective variety X, the following statements are equivalent.*

a) Let E be a curve semistable vector bundle over X. Then E is semistable with respect to some polarization H and $c_2(\mathrm{End}(E)) = 0 \in \mathrm{A}^2(X)$.

b) The Chern classes of any nflat vector bundle over X vanish.

Proof. If (a) holds, let E be a nflat vector bundle over X. By Lemma 3.3.8.b, E is semistable. Furthermore, applying also Lemma 3.3.3.a, E is curve semistable hence $c_2(E) = c_2(\mathrm{End}(E)) = 0$. By [42, Corollary 6], E is extension of vector bundles whose Chern classes vanish, hence the same vanishing holds for the Chern classes of E.

If (b) holds, let E be a curve semistable vector bundle over X. For any $f : C \to X$ one has $f^* \mathrm{End}(E) \cong \mathrm{End}(f^*E)$. By Lemma 2.3.1, $\mathrm{End}(E)$ is curve semistable. Since $\mathrm{End}(E)$ satisfies the hypotheses of Lemma 3.3.8.a, it is nflat hence $c_2(\mathrm{End}(E)) = 0$. By Lemmata 2.3.1 and 3.3.8.b, E is semistable. Q.e.d.

Remark 4.2.3. Since Theorem 4.2.1 proves that Theorem 4.2.2.a holds, one has another proof of Theorem 4.2.2.b. This has been proved by [19, Proposition 1.3] originally. $\diamondsuit$

In the Higgs bundles setting, Theorem 4.2.1 changes as it follows.

Theorem 4.2.4. *Let $\mathfrak{E} = (E, \varphi)$ a rank r Higgs bundle over a smooth projective variety X. Consider the following statements:*

a) $\theta_1(\mathfrak{E}), \ldots, \theta_{r-1}(\mathfrak{E})$ are nef;

b) $\mathfrak{E}$ is curve semistable;

c) $\mathfrak{E}$ *is semistable with respect to some polarization* H *and* $c_2(\mathrm{End}(E)) = 0 \in \mathrm{A}^2(X)$*;*

d) $\mathfrak{E}$ *is semistable with respect to some polarization* H *and* $\displaystyle\int_X c_2(\mathrm{End}(E)) \cdot H^{n-2} = 0$.

The following implications hold

$$(a) \Longleftrightarrow (b) \Longleftarrow (c) \Longleftrightarrow (d).$$

It is enough to repeat the proof of Theorem 4.2.1. However if $\mathfrak{E}$ is curve semistable then $\mathrm{End}(\mathfrak{E}) = (\mathrm{End}(E), \mathrm{End}(\varphi))$ is H-nflat (by Lemmata 2.3.1 and 3.3.8.a), hence $\mathfrak{E}$ is semistable, but it is unknown whether $c_2(\mathrm{End}(E)) = 0$.

Remark 4.2.5. If $\mathfrak{E}$ is semistable and $c_2(\mathrm{End}(E)) = 0$ then $\mathfrak{E}$ is semistable with respect to any polarization of X. This follows from the fact that the semistability of H-nflat Higgs bundles does not depend on the polarization of X. $\Diamond$

From all this, we are interested to study whether the condition 4.2.4.*b* implies the condition 4.2.4.*c*. To simplify the exposition of the corresponding topics, we introduce the following class

$$\Delta(E) = \frac{1}{2r} c_2(\mathrm{End}(E)) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 \in \mathrm{A}^2(X)$$

which is called *discriminant class of* E (cfr. Theorem 1.3.1). We recall the following conjecture.

Conjecture 1 (Bruzzo and Graña Otero Conjecture). Let $\mathfrak{E}$ be a curve semistable Higgs bundle over X. Then $\mathfrak{E}$ is semistable with respect to some polarization H and $\Delta(E) = 0$.

Remark 4.2.6. The best of our knowledge, the previous conjecture, assuming $\mathbb{K} = \mathbb{C}$, has been proved in the following cases:

a) $r = 2$, by [14, Theorems 4.5, 4.8 and 4.9];

b) X has nef tangent bundle, by [17, Corollary 3.15];

c) $\dim X = 2$ and $\kappa(X) \in \{-\infty, 0\}$ (the Kodaira dimension of X). Indeed, the statement for ruled surfaces follows by [17, Proposition 3.11]. Since rational surfaces are rationally connected, the statement follows by [17, Theorem 3.6]. The statement for Abelian surfaces follows by [17, Corollary 3.8], the case of K3 surfaces follows by [16, Theorem 6.4] and thus for Enriques surfaces and hyperelliptic surfaces follow by [17, Proposition 3.12]. All this complete this case;

d) $\dim X = 2$, $\kappa(X) = 1$ and other technical hypotheses, see [18, Proposition 5.6];

e) X is a simply-connected Calabi-Yau variety, by [9, Theorem 4.1];

f) if X satisfies the Conjecture 1 and Y is a fibred projective variety over X with rationally connected fibres, then Y does the same [17, Proposition 3.11];

g) if X satisfies the Conjecture 1 then any finite étale quotient Y of X does the same [17, Proposition 3.12];

h) $\mathfrak{E}$ has a JH-filtration whose quotient are H-nflat and have rank at most 2 (corollaries 4.3.6 and 4.3.7);

i) particular Higgs bundles described in [12], for more details see Appendix A. $\Diamond$

To be more precise, the Conjecture 1 can be simplified using [62, Lemma 3.7], as remarked in [44]. In other words, the Conjecture 1 can be rephrased as it follows.

Conjecture 2. Let $\mathfrak{E}$ be a curve semistable Higgs bundle over a smooth projective surface X. Then $\mathfrak{E}$ is semistable with respect to some polarization and $\Delta(E) = 0$.

Furthermore, the previous conjectures are equivalent to a third one.

Conjecture 3. Let $\mathfrak{E}$ be H-nflat over a smooth projective surface X. Then its Chern classes vanishes.

The fact that the previous conjecture implies Conjecture 2 is proved by [3, Corollary 3.2]. Furthermore, Biswas, Bruzzo and Gurjar have proved the other implication assuming $\mathbb{K} = \mathbb{C}$; here we give a proof which works under our assumption.

Proposition 4.2.7. *Conjecture 2 implies Conjecture 3.*

Proof. Let us assume $\mathfrak{E}$ is H-nflat and Conjecture 2 holds, by Lemmata 3.3.8.a and 3.3.8.b it is curve semistable with $\Delta(E) = c_2(E) = 0$ because $c_1(E) = 0$. Applying [42, Corollary 6], E is an iterating extension of vector bundles with vanishing Chern classes, hence the Chern classed of E vanish. Q.e.d.

Remark 4.2.8. When the Higgs field vanishes, [19, Proposition 1.3] proves the vanishing of Chern classes for nflat vector bundles over smooth projective varieties. This last result has been extended to compact Kähler manifolds by [20, Corollary 1.19]. $\Diamond$

4.3 Jordan-Hölder filtrations of H-nflat Higgs bundles

As done for Lemma 3.3.8.a, we improve Lemma 3.3.8.f generalising the hypothesis on the underlying field. We need some technical tools which are interesting by their own. To be exact, we extend some results known on the complex number field to any algebraically closed field of characteristic 0.

From now on, let $\mathfrak{E} = (E, \varphi)$ be a rank r Higgs bundle over a smooth projective polarized variety (X, H), defined over an algebraically closed field of characteristic 0, and let $\mathcal{E}$ be the sheaf of sections of E, if not otherwise indicated.

Lemma 4.3.1 (cfr. [20, Lemma 1.20])**.** *Let $\mathcal{F}$ be a rank s reflexive subsheaf of $\mathcal{E}$ such that the induced bundle morphism $\det(\mathcal{F}) \to \bigwedge^s \mathcal{E}$ is injective. Then $\mathcal{F}$ is locally free and it is a subbundle of E.*

Definition 4.3.2. A section s of E is *φ-invariant* if there exists a section λ of Ω^1_X such that $\varphi(s) = s \otimes \lambda$.

Proposition 4.3.3 ([9, Proposition 2.4])**.** *Let $\mathfrak{E} = (E, \varphi)$ be an H-nef Higgs bundle over X and let $\mathfrak{E}^\vee = (E^\vee, \varphi^\vee)$ be the dual Higgs bundle. If s is $\varphi^\vee$-invariant global section of $E^\vee$, then s has no zeroes.*

Proof. Note that s defines a monomorphism of Higgs sheaves $f\colon (\mathcal{O}_X, \lambda) \to (E^\vee, \varphi^\vee)$, where $\varphi^\vee(s) = s \otimes \lambda$ and $\lambda \in H^0(X, \Omega^1_X)$. Dualizing this monomorphism, one has a morphism of Higgs sheaves $f^\vee\colon (E, \varphi) \to (\mathcal{O}_X, \lambda)$; if s has zeroes, then $f^\vee$ has zeroes as well, and $\operatorname{Im} f^\vee$ is a proper Higgs subsheaf of $(\mathcal{O}_X, \lambda)$, hence it has negative degree on some projective curve in X. This contradicts the H-nefness of the Higgs quotient bundles of $\mathfrak{E}$ (see Lemmata 3.3.3.a and 3.3.3.b). Q.e.d.

Lemma 4.3.4 (cfr. [9, Lemma 3.1])**.** *Let $\mathfrak{E} = (E, \varphi)$ be an H-nflat Higgs bundle over X of rank $r \geq 2$. If $\mathfrak{E}$ is not stable, it can be written as an extension*

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{Q} \longrightarrow 0 \tag{4.3}$$

where $\mathfrak{F}$ and $\mathfrak{Q}$ are locally free H-nflat Higgs bundles, and $\mathfrak{F}$ is stable.

Proof. Note that $\mathfrak{E}$ is semistable by Lemma 3.3.8.b and has degree zero. Let $\mathfrak{F} = (F, \psi)$ be a Higgs subsheaf of E of rank p, with $0 < p < r$. As $\mathfrak{E}$ is semistable of zero degree, $\bigwedge^p \mathfrak{E}$ is semistable of zero degree as well by Theorem 2.2.5. Let $\det(F) = \left(\bigwedge^p F\right)^{\vee\vee}$ be the determinant of F, and let $\det(\mathfrak{F})$ be the sheaf $\det(F)$ equipped with the naturally induced Higgs field. As $\det(\mathfrak{F})$ injects into $\bigwedge^p \mathfrak{E}$ (as a Higgs sheaf), we have $\deg(F) \leq 0$.

We can assume that $\mathfrak{F}$ is a reflexive Higgs subsheaf of $\mathfrak{E}$ of minimal rank $p > 0$ with $\deg(F) = 0$. Then $\mathfrak{F}$ is stable. We have an exact sequence

$$0 \longrightarrow \det(\mathfrak{F}) \longrightarrow \bigwedge^p \mathfrak{E} \longrightarrow \mathfrak{R} \longrightarrow 0 \tag{4.4}$$

where $\mathfrak{R} = (R, \chi)$ is the quotient Higgs sheaf. We use this to show that $(\det(F))^\vee$ is nef. Let $f\colon C \to X$ be a morphism, where C is a smooth irreducible projective curve. Then f^*R splits as $\widetilde{R} \oplus T$, where $\widetilde{R}$ is a locally free sheaf and T is a torsion sheaf; in particular, T with the restriction of the pullback Higgs field is a Higgs sheaf[2]. Thus $\widetilde{R}$, again with the restriction of the pullback Higgs field, is a Higgs quotient bundle of f^*R hence it is a locally free quotient of $f^*\left(\bigwedge^p \mathfrak{E}\right)$, therefore it is H-nef, and then $\deg(f^*R) \geq 0$. Then $\deg(f^*\det(F)) \leq 0$, and since the choice of C is arbitrary, $(\det(F))^\vee$ is nef.

Since $c_1(F) \equiv_{num} 0$, $\det(F)$ is numerically flat. Tensoring the exact sequence (4.4) by $\det^{-1}\mathfrak{F}$ one obtains a $\det\psi^\vee \otimes \varphi^p$-invariant section $\sigma\colon (\mathcal{O}_X, \lambda) \to (\det(\mathfrak{F}))^\vee \otimes \bigwedge^p \mathfrak{E}$, where φ^p is the Higgs field of $\bigwedge^p \mathfrak{E}$ and $(\det\psi^\vee \otimes \varphi^p)(\sigma) = \sigma \otimes \lambda$. By Proposition 4.3.3, σ has no zeroes, *i.e.* $\det(\mathfrak{F})$ is a Higgs subbundle of $\bigwedge^p \mathfrak{E}$; by Lemma 4.3.1 $\mathfrak{F}$ is a Higgs subbundle of $\mathfrak{E}$. Thus $\mathfrak{F}$ is an H-nflat Higgs bundle, and then by Lemma 3.3.8.e the quotient Higgs sheaf $\mathfrak{Q}$ is locally free and H-nflat as well. Q.e.d.

Theorem 4.3.5 (cfr. [9, Thereom 3.2]). *Let $\mathfrak{E}$ be a Higgs bundle over X. $\mathfrak{E}$ is H-nflat if and only if it is* pseudostable *(*i.e.*, it has a filtration whose quotients are locally free and stable), and the quotients of the filtration are H-nflat.*

Proof. Assuming that such a filtration exists, then $\mathfrak{E}$ is H-nflat by Lemma 3.3.8.c.

Vice versa, let $\mathfrak{E}$ be H-nflat. We use Lemma 4.3.4 as the basis for an iterative proof. Note that in eq. (4.3) if the Higgs bundle $\mathfrak{Q}$ is stable, we have the claim. Otherwise, $\mathfrak{Q}$ satisfies the same hypothesis as $\mathfrak{E}$, so that it sits in a short exact sequence

$$0 \longrightarrow \mathfrak{Q}_1 \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{Q}_2 \longrightarrow 0$$

[2]See footnote 2 at page 5.

where $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are locally free and H-nflat and $\mathfrak{Q}_1$ is stable. By the Snake Lemma we have a diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \mathfrak{Q}_1 \arrow[d] & \\
0 \arrow[r] & \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d] & \mathfrak{Q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{F}_1 \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d] & \mathfrak{Q}_2 \arrow[r] \arrow[d] & 0 \\
 & \mathfrak{Q}_1 \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Note that again $\mathfrak{F}_1$ is locally free and H-nflat by Lemma 3.3.8.e. Therefore $0 \subsetneqq \mathfrak{F} \subsetneqq \mathfrak{F}_1 \subsetneqq \mathfrak{E}$ is a filtration whose quotients $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are locally free and H-nflat; moreover, $\mathfrak{F}$ and $\mathfrak{Q}_1$ are stable. If $\mathfrak{Q}_2$ is stable as well, the claim is proved. If it is not, we iterate the procedure, until we get a locally free quotient which is stable (possibly a line bundle). At step k we shall have the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \mathfrak{Q}_k \arrow[d] & \\
0 \arrow[r] & \mathfrak{F}_{k-1} \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d] & \mathfrak{Q}_{k-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{F}_k \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d] & \mathfrak{Q}_{k+1} \arrow[r] \arrow[d] & 0 \\
 & \mathfrak{Q}_k \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

and if m is the last step we get a filtration

$$0 = \mathfrak{F}_0 \subsetneqq \mathfrak{F} \subsetneqq \mathfrak{F}_1 \subsetneqq \ldots \subsetneqq \mathfrak{F}_m \subsetneqq \mathfrak{F}_{m+1} = \mathfrak{E} \tag{4.5}$$

whose quotients $\mathfrak{F}, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_{m+1}$ are locally free, stable and H-nflat. Q.e.d.

By the previous theorem, we can prove Conjecture 3 in new cases. From now on $\dim X = 2$.

Corollary 4.3.6 (cfr. [9, Corollary 3.3]). *If $\mathfrak{E} = (E, \varphi)$ is an H-nflat Higgs bundle over X such that all the quotients of the filtration (4.5) have rank 1*[3]*, then $\mathfrak{E}$ has vanishing Chern classes.*

Proof. Indeed $c_1(Q_k) = 0$ for all $k \geq 1$ as each $\mathfrak{Q}_k = (Q_k, \widetilde{\varphi}_k)$ is an H-nflat line bundle, so that $c_h(E) = 0$ for all $h \in \{1, \ldots, \min\{n, r\}\}$. Q.e.d.

Corollary 4.3.7. *If $\mathfrak{E} = (E, \varphi)$ is an H-nflat Higgs bundle over X such that all the quotients of the filtration (4.5) have rank at most 2, then $\mathfrak{E}$ has vanishing Chern classes.*

Proof. Each $\mathfrak{Q}_k$ is either an H-nflat line bundle or a rank 2 H-nflat Higgs bundle, and this last case $\mathfrak{Q}_k = (Q_k, \widetilde{\varphi}_k)$ has vanishing Chern classes. Indeed, if $\widetilde{\varphi}_k \neq 0$ then $\mathfrak{Gr}_1(\mathfrak{Q}_k)$ has an irreducible component Z which is a divisor of $\mathrm{Gr}_1(Q_k)$ and surjects onto X (Proposition 3.1.4), because

$$2 = \dim X \leq \dim Z < \dim \mathrm{Gr}_1(Q_k) = 3,$$

hence $c_2(Q_k) = 0$ by [14, Theorem 3.3]. Otherwise, if $\widetilde{\varphi}_k = 0$ then $c_2(Q_k) = 0$ by [19, Proposition 1.3]. From all this, $c_h(E) = 0$ for all $h \in \{1, \ldots, \min\{n, r\}\}$. Q.e.d.

Remark 4.3.8. Let $\mathfrak{E}$ be a rank 3 H-nflat not stable Higgs bundle over X. By Lemma 4.3.4 and previous corollary, its Chern classes vanish. $\Diamond$

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[3] Heuristically, these are the H-nflat Higgs bundles that are the farthest from being stable.

Chapter 5

The Simpson System

In this chapter, we give application of the positivity conditions for Higgs bundles and the theory of curve semistable Higgs bundles to minimal smooth projective varieties of general type.

Whenever we consider a morphism $f\colon C \to X$, we understand that C is an irreducible smooth projective curve.

5.1 The Simpson system on minimal smooth projective surfaces of general type

The contents of this section are mainly based on the paper [10] written in collaboration with Ugo Bruzzo and Beatriz Graña Otero, unless otherwise indicated.

Let X be a minimal smooth surface of general type. By [24, Proposition 10.7], the canonical bundle K_X of X is nef as well as being big. The Chern classes of X satisfy the *Bogomolov-Miyaoka-Yau inequality* (*BMY-inequality*, for short)

$$BMY(X) \overset{def.}{=} \int_X 3c_2(X) - c_1(X)^2 \geq 0, \tag{5.1}$$

where $c_k(X) \overset{def.}{=} c_k(TX)$ for any k. This has been proved in [69, Theorem 4] and [50, Theorem 4] over $\mathbb{C}$ and in [52, Proposition 7.1] over $\mathbb{K}$, an algebraically closed field of characteristic 0. Moreover Miyaoka, working over $\mathbb{C}$, has proved in [51, Corollary in the Appendix in Paragraph 2] that the cotangent bundle of a surface of general type that saturates the inequality (5.1) is ample.

In this section we extend this result over $\mathbb{K}$ relying on the H-ampleness and H-nefness criteria as expressed by [3, Lemma 3.3] and Theorem 3.2.6.

This will be based on the properties of the so called *Simpson system*, *i.e.* the Higgs bundle $\mathfrak{S} = (S, \varphi)$, where $S = \Omega^1_X \oplus \mathcal{O}_X$ and

$$\varphi = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \end{pmatrix}, \ \mathrm{Id} \in \mathrm{Hom}\left(\Omega^1_X, \Omega^1_X\right).$$

In the complex setting, Simpson proved that $\mathfrak{S}$ is stable with respect to a polarization H if some inequalities on the Chern classes of X hold (see [61, Proposition 9.9]). In particular, these inequalities are satisfied by the minimal smooth surfaces of general type which saturate the inequality (5.1). On the other hand, Langer has extended this result in [42, Proposition 3 and Remark 4], when $\mathbb{K}$ is an algebraically closed field of any characteristic.

From now on, X is a minimal smooth surface of general type over $\mathbb{K}$ which saturates the inequality (5.1), unless otherwise indicated. These projective surfaces exist over $\mathbb{C}$ as Miyaoka has proved in [50, Theorem 5]. They satisfy some interesting properties. Here we give new proofs of some properties of these surfaces using the H-ampleness and H-nefness of the twisted Simpson systems over them. These proofs are based on the curve semistability of $\mathfrak{S}$; this fact extends the previous results of Simpson and Langer.

Proposition 5.1.1. *Let X be a minimal smooth surface of general type over $\mathbb{K}$ such that $BMY(X) = 0$. Then the Higgs bundle $\mathfrak{S}$ is curve semistable.*

Proof. By [42, Proposition 3 and Remark 4], $\mathfrak{S}$ is stable. Since

$$\int_X \Delta(X) = \int_X c_2(\Omega_X) - \frac{1}{3}c_1(\Omega_X)^2 = \int_X c_2(X) - \frac{1}{3}3c_2(X) = 0,$$

hence the statement follows from Theorem 4.2.1. Q.e.d.

Remark 5.1.2. On smooth complex projective surfaces with ample canonical bundle, $\mathfrak{S}$ is stable: see Proposition 5.1.10. $\Diamond$

Theorem 5.1.3. *Let X be a minimal smooth surface of general type over $\mathbb{K}$ such that $BMY(X) = 0$. Then the Higgs bundle $\mathfrak{S}_\beta = \mathfrak{S}(-\beta K_X)$ is H-nef for every rational number*[1] *$\beta \leq \dfrac{1}{3}$.*

[1] As it is customary, we formally consider twistings by rational divisors, which make sense after pulling back to a (possibly ramified) finite covering of X; on the other hand, the properties of being semistable, H-ample, H-nef are invariant under such coverings (see Proposition 3.2.5.a, Lemmata 3.3.3.a and 4.1.5).

Proof. By Propositions 1.2.8 and 5.1.1 and by Lemma 4.1.5 $\mathfrak{S}_\beta$ is curve semistable, so that for every morphism $f\colon C \to X$, the pullback Higgs bundle $f^*\mathfrak{S}_\beta$ is semistable, hence

$$\mu\left(f^*\mathfrak{S}_\beta\right) = \mu_{\min}\left(f^*\mathfrak{S}_\beta\right) = \frac{1}{3}\int_C f^*c_1(S_\beta) = \frac{1}{3}\int_{C'} c_1(S(-\beta K_X)) = \left(\frac{1}{3}-\beta\right)\int_{C'} K_X \geq 0$$

where $C' = f(C)$; the last inequality holds as K_X is nef. By Theorem 1.4.9 any Higgs quotient of $f^*\mathfrak{S}_\beta$ has non negative degree. Furthermore, by the previous computation

$$\int_C f^*c_1(S_\beta) = (1-3\beta)\int_{C'} K_X \geq 0.$$

Thus, by the H-nefness criterion (Corollary 3.3.5), we have the nefness of $\det(\mathfrak{S}_\beta)$, so that the claim follows. Q.e.d.

Corollary 5.1.4 (cfr. [51, Proposition 5 and Corollary in the Appendix to Paragraph 2])**.** *Let X be a minimal smooth surface of general type over $\mathbb{K}$ such that $BMY(X) = 0$. Then the vector bundle $\Omega_\beta = \Omega^1_X(-\beta K_X)$ is nef for every rational number $\beta \leq \frac{1}{3}$. As a consequence:*

a) K_X is ample;

b) any projective curve C on X satisfies the inequality $2p_a(C) - 2 \geq \frac{1}{3}\int_X K_X \cdot C$ where $p_a(C)$ is the arithmetic genus *of C. In particular X does not contain neither rational curves nor curves of arithmetic genus 1.*

Proof. Ω_β with the zero Higgs field is a Higgs quotient of $\mathfrak{S}_\beta$, hence it is H-nef (Lemma 3.3.3.b) and then nef in the usual sense (Remark 3.3.2.a). Let $C \subset X$ be an irreducible projective curve, let $\left(\widetilde{C}, \nu\right)$ be its normalization, and let $\iota\colon \widetilde{C} \xrightarrow{\nu} C \hookrightarrow X$. We have the right exact sequence

$$\iota^*\Omega_{\frac{1}{3}} \longrightarrow \Omega^1_{\widetilde{C}}\left(-\frac{1}{3}\iota^*K_X\right) = \mathcal{O}_{\widetilde{C}}\left(K_{\widetilde{C}} - \frac{1}{3}\iota^*K_X\right) \longrightarrow 0$$

so that $\mathcal{O}_{\widetilde{C}}\left(K_{\widetilde{C}} - \frac{1}{3}\iota^*K_X\right)$ is nef, *i.e.* its degree is nonnegative, and

$$2p_a(C) - 2 \geq 2p_a\left(\widetilde{C}\right) - 2 = \deg \Omega^1_{\widetilde{C}} \geq \frac{1}{3}\int_X K_X \cdot C \geq 0.$$

As a consequence X has no rational curves, then by [6, Proposition 1], $\int_X K_X \cdot C > 0$. Since this inequality holds for any projective curve on X and $\int_X K_X^2 > 0$ ([45, Theorem 2.2.16]), it follows that K_X is ample by Nakai-Moišezon Criterion ([30, Theorem V.1.10]). Q.e.d.

On the other hand, the previous inequality is not sharp as the following example proves.

Example 5.1.5. Let X be a *fake complex projective plane*, *i.e.* X is a smooth projective surface with the same *Betti numbers of* $\mathbb{P}^2_{\mathbb{C}}$ but is not isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ ([54], [36, Section 5] and [37, Theorem 3.1]). Its canonical bundle K_X is ample hence X is a minimal smooth surface of general type and $\displaystyle\int_X 3c_2(X) = \int_X c_1(X)^2 = 9$ and the *Picard number* $\rho(X)$ *of* X *is* 1.

Let $\mathcal{O}_X(1)$ be the ample generator of the torsion-free part of $\operatorname{Pic}(X)$ such that $K_X = \mathcal{O}_X(3)$ and $c_1(\mathcal{O}_X(1))^2 = 1$ (cfr. [22, Section 1.1]). Moreover, let $f\colon C \to X$ non-constant then $g(C) \geq 3$ by [38, Lemma 2.2] and [22, Proposition 2.3]. However, for a such C the previous corollary predicts:

$$2g(C) - 2 \geq \frac{1}{2+1}\int_X \mathcal{O}_X(3)C = \int_X \mathcal{O}_X(1)C > 0 \iff g(C) \geq 2.$$

$\triangle$

Remark 5.1.6 (cfr. [51, Corollary in the Appendix to Paragraph 2])**.** Now

$$\Omega^1_X \simeq \Omega^1_X\left(-\frac{1}{3}K_X\right) \otimes \mathcal{O}_X\left(\frac{1}{3}K_X\right)$$

by the previous corollary, Ω^1_X is the tensor product of a nef bundle by an ample line bundle, and therefore is ample by [45, Corollary 1.4.10]. $\Diamond$

It may be instructive to deduce the previous remark from the H-ampleness criterion of Theorem 3.2.6.

Lemma 5.1.7. *Let X be a minimal smooth surface of general type over $\mathbb{K}$ such that $BMY(X) = 0$. Then the Higgs bundle $\mathfrak{S}$ (the Simpson system) is H-ample.*

Proof. Fix an ample class h. By the H-ampleness criterion (Theorem 3.2.6) a Higgs bundle $\mathfrak{E}$ is H-ample if and only if its determinant is ample and there exists a $\delta \in \mathbb{R}_{>0}$ such that

$$\mu_{\min}(f^*\mathfrak{E}) \geq \delta \int_C f^*h$$

for all $f\colon C \to X$. Note that $\det(S) = K_X$ is ample by Corollary 5.1.4. We take $h = K_X$ and $\delta = \dfrac{1}{3}$. Since $\mathfrak{S}$ is curve semistable (Proposition 5.1.1)

$$\mu_{\min}(f^*\mathfrak{S}) = \mu(f^*\mathfrak{S}) = \frac{1}{3}\int_C f^*c_1(S) = \delta \int_C f^*K_X.$$

Q.e.d.

Remark 5.1.8. S, as an ordinary bundle, is neither stable nor ample. Indeed, $\mathcal{O}_X$ is a quotient of S and $\mu(S) > \mu(\mathcal{O}_X) = 0$ (Remark 3.2.4.a, Propositions 1.2.5 and 3.2.5.b). $\Diamond$

Now we are in position to give the example announced in Remark 5.1.2.

Example 5.1.9 (cfr. [15, Example 3.9])**.** Let X be a minimal smooth surface of general type such that $BMY(X) > 0$, Ω^1_X is nef but not ample and K_X is ample[2]. For simplicity we assume $\mathbb{K} = \mathbb{C}$. By [65, Theorem 1] and Lemma 1.2.9, Ω^1_X is semistable with respect to the polarization K_X.

By Bertini's Theorem (cfr. [30, Corollary III.10.9 and Exercise III.11.3]), for $m \gg 1$ there exists a smooth irreducible projective curve $C \in |mK_X|$, and by [49, Theorem 6.1] $\Omega^1_{X|C}$ is semistable. Let us consider $\mathfrak{Gr}_1(\mathfrak{S}_{|C})$; it coincides with $\mathrm{Gr}_1(\Omega^1_C) \cong C$. Indeed, consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r, hook, "\eta"] \arrow[d, "0"] & \mathfrak{S}_{|C} \arrow[r, two heads, "\epsilon"] \arrow[d, "\varphi_{|C}"] & \Omega^1_{X|C} \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \Omega^1_C \arrow[r, hook, "\eta\otimes\mathrm{Id}"'] & \mathfrak{S}_{|C}\otimes\Omega^1_C \arrow[r, two heads, "\epsilon\otimes\mathrm{Id}"'] & \Omega^1_{X|C}\otimes\Omega^1_C \arrow[r] & 0
\end{tikzcd}$$

which is commutative by construction. Computing the Higgs field ψ induced on $\Omega^1_{X|C}$, we reason on the stalks. We have

$$\forall x \in C, \widetilde{\omega} \in \Omega^1_{X|C},\ \epsilon_x^{-1}(\widetilde{\omega}) = \{(s,\omega) \in \mathfrak{S}_x \mid \pi(\omega) = \widetilde{\omega}\},$$
$$\varphi_{|C,x}\left(\epsilon_x^{-1}(\widetilde{\omega})\right) = \{\widetilde{\omega}\} \subseteq \Omega^1_{C,x} = \ker(\epsilon\otimes\mathrm{Id})_x$$

where $\pi\colon \Omega^1_{X|C} \to \Omega^1_C$ is the canonical projection. Thus ψ vanishes. On the other hand:

$$\forall x \in C,\ \ker\left(\varphi_{|C,x}\right) = \{(\omega_x, f_x) \in S_x \mid \pi(\omega_x) = 0\} \cong \ker(\pi)_x \oplus \mathcal{O}_{C,x} \Rightarrow \ker\left(\varphi_{|C}\right) \cong \ker(\pi) \oplus \mathcal{O}_C,$$
$$\mathrm{Im}\left(\varphi_{|C,x}\right) \cong \mathfrak{S}_x / \ker\left(\varphi_{|C,x}\right) \cong \Omega^1_{C,x} \Rightarrow \mathrm{Im}\left(\varphi_{|C}\right) \cong \Omega^1_C,$$

thus $\ker(\pi) = \mathcal{N}^\vee_{C/X}$ (the *conormal bundle of C in X*). It is a locally free sheaf because C is a smooth projective curve. From all this, $(\Omega^1_C, 0)$ is a rank 1 Higgs quotient bundle of $\left(\Omega^1_{X|C}, 0\right)$ hence $\mathrm{Gr}_1(\Omega^1_C) \subseteq \mathrm{Gr}_1\left(\Omega^1_{X|C}\right)$.

[2]Following [59, Example 1.7], let A be an Abelian 3-fold containing an elliptic curve E, let X be a sufficiently positive smooth divisor of A containing E. Then X is a surface of general type; indeed, by Adjunction Formula $K_X = X_{|X}$ and by assumption this is big. Since A contains no rational curves, the same is true for X, therefore X is a minimal model ([30, Theorem V.5.7]) and K_X is ample (cfr. proof of Corollary 5.1.4). Finally, Ω^1_X is a quotient of $\Omega^1_{A|X} = \mathcal{O}_X^{\oplus 3}$, hence it is nef. Since X contains the elliptic curve E, then $\displaystyle\int_X 3c_2(X) - c_1(X)^2 > 0$ (Corollary 5.1.4) and by [45, Example 6.3.28] Ω^1_X is not ample.

Let $\mathfrak{K} = \left(\mathcal{K}, \varphi_{|C,\mathcal{K}}\right)$ be a rank 2 Higgs subsheaf of $\mathfrak{S}_{|C}$ such that the corresponding Higgs quotient $\mathfrak{Q} = \left(Q, \widetilde{\varphi}_{|C}\right)$ is locally free. By definition $\varphi_{|C}(\mathcal{K}) = \underline{0}_C \oplus \pi(\mathcal{K}) \subseteq \mathcal{K} \otimes \Omega^1_C$; this is possible if and only if $\mathcal{K} = \ker(\pi) \oplus \mathcal{O}_C$. From this follows that $(\Omega^1_C, 0)$ is the only rank 1 Higgs quotient bundle of $\mathfrak{S}_{|C}$, *i.e.* $\mathfrak{Gr}_1(\mathfrak{S}_{|C}) = \mathrm{Gr}_1\left(\Omega^1_C\right)$.

So

$$\int_{\mathrm{Gr}_1\left(\Omega^1_C\right)} \theta_1\left(\mathfrak{S}_{|C}\right) = \int_C c_1\left(\Omega^1_C\right) - \frac{1}{3} c_1\left(S_{|C}\right) =$$

$$= \int_X C \cdot C + C \cdot K_X - \frac{1}{3}\int_X C \cdot K_X = \left(m^2 + \frac{2}{3}m\right)\int_X K_X^2 > 0$$

i.e. $\theta_1\left(\mathfrak{S}_{|C}\right)$ is positive.

Let $\mathfrak{L} = \left(\mathcal{L}, \varphi_{|C,\mathcal{L}}\right)$ be a rank 1 Higgs subsheaf of $\mathfrak{S}_{|C}$ such that the corresponding Higgs quotient sheaf $\mathfrak{Q} = (\mathcal{Q}, \widetilde{\varphi})$ is locally free. If $\mathcal{L} \subseteq \mathcal{O}_C$ then we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{L} \arrow[r, hook] \arrow[d, equal] & \mathcal{O}_C \arrow[r, two heads] \arrow[d, hook] & \mathcal{Q}_0 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r, hook] \arrow[d] & S_{|C} \arrow[r, two heads] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \Omega^1_{X|C} \arrow[r, equal] \arrow[d] & \Omega^1_{X|C} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $\mathrm{rank}\left(\mathcal{Q}_0\right) = 0$ and $\mathcal{Q} = \Omega^1_{X|C} \oplus \mathcal{Q}_0$. Since $\mathcal{Q}$ is torsion-free then $\mathcal{Q}_0 = \underline{0}_C$, hence $\mathcal{L} = \mathcal{O}_C$. On the other hand, let $\mathcal{L} \subsetneq \Omega^1_{X|C}$ such that $\varphi_{|C}(\mathcal{L}) \subseteq \mathcal{L} \otimes \Omega^1_C$. This last condition does not happen, because $\varphi_{|C}(\mathcal{L}) \subseteq \underline{0}_C \oplus \Omega^1_C \subseteq (\Omega^1_C \otimes \Omega^1_C) \oplus \Omega^1_C$, that is $\Omega^1_{X|C}$ does not contain Higgs subsheaves of $\mathfrak{S}_{|C}$.

Thus $\mathfrak{Q}$ is $\left(\Omega^1_{X|C}, 0\right)$ hence $C \cong \mathfrak{Gr}_2\left(\mathfrak{Q}\right) = \mathfrak{Gr}_2\left(\mathfrak{S}_{|C}\right)$ is irreducible. And

$$\int_{\mathrm{Gr}_2\left(\Omega^1_{X|C}\right)} \theta_2\left(\mathfrak{S}_{|C}\right) = \int_{\mathrm{Gr}_2\left(\Omega^1_{X|C}\right)} c_1\left(\Omega^1_{X|C}\right) - \frac{2}{3} c_1\left(S_{|C}\right) =$$

$$= \frac{1}{3}\int_{\mathrm{Gr}_2\left(\Omega^1_{X|C}\right)} c_1\left(\Omega^1_{X|C}\right) = \frac{m}{3}\int_X K_X^2 > 0,$$

by definition $\theta_2\left(\mathfrak{S}_{|C}\right)$ is positive.

From all this, $\mathfrak{S}_{|C}$ is semistable (Theorem 4.1.2). Moreover we have proved

$$\mu\left(S_{|C}\right) < \mu\left(\Omega^1_{X|C}\right), \ \mu\left(S_{|C}\right) < \mu\left(\Omega^1_C\right)$$

that is $\mathfrak{S}_{|C}$ is stable hence $\mathfrak{S}$ is stable with respect to K_X.

Note that Ω^1_X is not curve semistable (Theorem 4.2.1), *i.e.* there exists $f_0\colon C_0 \to X$ such that $\Omega^1_{X|C_0} \equiv f_0^*\Omega^1_X$ is unstable. △

Implicitly, the previous example proves the following proposition.

Proposition 5.1.10. *Let X be a smooth complex projective surface such that K_X is ample. Then $\mathfrak{S}$ is stable with respect to K_X.*

5.2 The Simpson system on minimal smooth complex projective varieties of general type

From now on, let X be a minimal smooth complex projective variety of general type with $n \geq 2$ and K_X is ample, let $\mathfrak{S} = (S, \varphi)$ be the Simpson system, and let

$$GY(X) \stackrel{def.}{=} (-1)^n \int_X \left(c_2(X) - \frac{n}{2(n+1)} c_1(X)^2\right) \cdot c_1(X)^{n-2}. \tag{5.2}$$

Simpson has asked in [61] how to go between the condition $GY(X) = 0$, the stability of $\mathfrak{S}$ and the ampleness of K_X when $n \geq 3$. Here we give a solution to this problem.

Theorem 5.2.1. *Let X be a smooth complex projective variety such that K_X is ample. The Higgs bundle $\mathfrak{S}$ is stable with respect to K_X.*

Proof. As in Example 5.1.9, Ω^1_X is semistable with respect to K_X.

By Bertini's Theorem (cfr. [30, Corollary III.10.9 and Exercise III.11.3]), for $m \gg 1$ there exist ample smooth divisors $D_1, \ldots, D_{n-1} \in |mK_X|$ such that $Y_p = D_1 \cap \ldots \cap D_p$ are smooth projective subvarieties of X for any $p \in \{1, \ldots, n-1\}$[3]; we put $C = Y_{n-1}$. By [49, Theorem 6.1] $\Omega^1_{X|C}$ is semistable.

The Higgs field induced by $\varphi_{|C}$ on the Higgs quotient $\Omega^1_{X|C}$ vanishes (cfr. Example 5.1.9).

[3]Moreover, these smooth projective varieties are all minimal and of general type. Indeed, by Adjunction Formula $K_{Y_1} = (K_X + Y_1)_{|Y_1}$ ([66, Exercise 21.5.B]) and this is an ample line bundle ([45, Proposition 1.2.13 and Corollary 1.4.10]). Iterating these reasoning for all Y_p one has the claim.

We have the following inequality

$$0 < \mu\left(S_{|C}\right) = \frac{1}{n+1}\int_C c_1\left(S_{|C}\right) < \frac{1}{n}\int_C c_1\left(\Omega^1_{X|C}\right) = \mu\left(\Omega^1_{X|C}\right).$$

Let $\varpi \colon \mathfrak{S}_{|C} \to \mathfrak{Q} = (\mathcal{Q}, \widetilde{\varphi})$ be a rank r Higgs quotient bundle of $\mathfrak{S}$ and let $\mathcal{K} = \ker(\varpi)$; $\mathcal{K}$ is locally free because it is a kernel of an epimorphism of locally free sheaves on a Noetherian scheme. We set

$$\mathcal{K}_1 = \mathcal{K} \cap \Omega^1_{X|C}, \mathcal{K}_2 = \mathcal{K} \cap \mathcal{O}_C.$$

1) Let $\mathcal{K}_2 = \underline{0}_C$ then $\mathcal{K} \subseteq \Omega^1_{X|C}$. This cannot happen, indeed, by assumption $\varphi_{|C}(\mathcal{K}) \subseteq \mathcal{K} \otimes \Omega^1_C$ but by definition $\varphi_{|C}(\mathcal{K}) = \underline{0}_C \oplus \pi(\mathcal{K}) \subseteq \left(\Omega^1_{X|C} \otimes \Omega^1_C\right) \oplus \Omega^1_C$, that is $\mathcal{K}$ cannot be a Higgs subsheaf of $\mathfrak{S}_{|C}$. In other words, $\Omega^1_{X|C}$ does not contain Higgs subsheaves of $\mathfrak{S}_{|C}$.

2) Let $\mathcal{K}_2 \neq \underline{0}_C$ then $\mathcal{K}_2 = \mathcal{O}_C$. Indeed, consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{K}_2 \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_C \arrow[d, hook] & & \\
0 \arrow[r] & \mathcal{K} \arrow[r, hook] \arrow[d] & S_{|C} \arrow[r, two heads] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \mathcal{K}_1 \arrow[r, hook] \arrow[d] & \Omega^1_{X|C} \arrow[r, two heads] \arrow[d] & \mathcal{Q}_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns and the rows are short exact sequence of sheaves. By the universal property of cokernels of morphisms, there exists a unique morphism $q \colon \mathcal{Q} \twoheadrightarrow \mathcal{Q}_0$ which makes commutative the diagram, and it is also an epimorphism. Computing the ranks:

$$\operatorname{rank}\left(S_{|C}\right) = n+1, \operatorname{rank}(\mathcal{Q}) = r \Rightarrow \operatorname{rank}(\mathcal{K}) = n - r + 1$$
$$\operatorname{rank}\left(\mathcal{K}_2\right) = 1, \operatorname{rank}(\mathcal{K}) = n - r + 1 \Rightarrow \operatorname{rank}\left(\mathcal{K}_1\right) = n - r$$
$$\operatorname{rank}\left(\Omega^1_{X|C}\right) = n, \operatorname{rank}\left(\mathcal{K}_1\right) = n - r \Rightarrow \operatorname{rank}\left(\mathcal{Q}_0\right) = r$$
$$\operatorname{rank}(\mathcal{Q}) = \operatorname{rank}\left(\mathcal{Q}_0\right) = r \Rightarrow \operatorname{rank}\left(\ker(q)\right) = 0;$$

on the other hand, $\ker(q)$ is locally free, because it is a torsion-free sheaf on a smooth curve ([57, Corollary at page 75]). Thus $\ker(q) = \underline{0}_C$ and $\mathcal{Q} \cong \mathcal{Q}_0$. Applying the Snake Lemma, we have the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2 \arrow[r, hook] & \mathcal{O}_C \arrow[r] & \ker(q) = \underline{0}_C \end{tikzcd}$$

i.e. $\mathcal{K}_2 = \mathcal{O}_C$. From all this, it turns out that $\mathfrak{Q}$ is a Higgs quotient bundle of $\left(\Omega^1_{X|C}, 0\right)$. Thus $\mu\left(S_{|C}\right) < \mu\left(\Omega^1_{X|C}\right) \leq \mu(Q)$ because $\Omega^1_{X|C}$ is semistable.

From all this, the claim follows from Proposition 1.2.5. Q.e.d.

Corollary 5.2.2. *Let X be a smooth complex projective variety such that K_X is ample. If $GY(X) = 0$ then*

a) the Higgs bundle $\mathfrak{S}$ is curve semistable;

b) $\Delta(S) = c_2(X) - \dfrac{n}{2(n+1)} c_1(X)^2 = 0 \in \mathrm{H}^4(X, \mathbb{Q})$*;*

c) the Higgs bundle $\mathfrak{S}$ is semistable with respect to any polarization of X.

Proof. (a) and (b). These follow directly from Theorem 4.2.4.

However we give another proof: since $(\Omega^1_X, 0)$ is a rank n Higgs quotient bundle of $\mathfrak{S}$, then $\mathrm{Gr}_n\left(\Omega^1_X\right) \subseteq \mathfrak{Gr}_n(\mathfrak{S})$ and

$$\dim \mathrm{Gr}_n\left(\Omega^1_X\right) = \operatorname{rank}\left(\Omega^1_X\right) + \dim X - 1 = 2n - 1 = \dim \mathfrak{Gr}_n(\mathfrak{S}) < \dim \mathrm{Gr}_n(S) = 2n,$$

hence $\mathfrak{Gr}_n(\mathfrak{S})$ has an irreducible component which is a divisor of $\mathrm{Gr}_n(S)$ and surjects onto X. Thus $\mathfrak{S}$ is curve semistable by [14, Theorem 4.9], because $\mathfrak{S}$ is semistable and $\Delta(S) = 0$.

(c). This follows by Remark 4.2.5. Q.e.d.

This last corollary permits us to extend Lemma 5.1.7 to any higher dimension; it is enough to mimic the proof of dimension 2 case.

Lemma 5.2.3. *Let X be a smooth complex projective variety such that K_X is ample and $GY(X) = 0$. Then the Higgs bundle $\mathfrak{S}$ is H-ample.*

By the proof of Theorem 5.2.1, applying Proposition 3.2.5.b, we have the following corollary.

Corollary 5.2.4. *Let X be a smooth complex projective variety such that K_X is ample and $GY(X) = 0$. Then Ω^1_X is ample.*

Remark 5.2.5. S, as an ordinary bundle, is neither stable nor ample (cfr. Remark 5.1.8). $\diamondsuit$

On the other hand, we have another proof of *Guggenheimer-Yau inequality* ([69, Remark (iii)]).

Theorem 5.2.6. *Let X be a smooth complex projective variety such that K_X is ample. Then $GY(X) \geq 0$.*

Proof. It is enough to note that $GY(X) = \int_X \Delta(S) \cdot c_1(K_X)^{n-2}$ and the claim follows from Theorem 1.3.1. Q.e.d.

Thanks to the previous corollary we can simplify [10, Proposition 4.5].

Theorem 5.2.7. *Let X be a smooth complex projective variety such that K_X is ample and $GY(X) = 0$. Then the Higgs bundle $\mathfrak{S}_\beta = \mathfrak{S}(-\beta K_X)$ is H-nef for every rational number $\beta \leq \dfrac{1}{n+1}$.*

It is enough to mimic the proof of Theorem 5.1.3.

Corollary 5.2.8. *Let X be a smooth complex projective variety such that K_X is ample and $GY(X) = 0$. Then the vector bundle $\Omega_\beta = \Omega^1_X(-\beta K_X)$ is nef for every rational number $\beta \leq \dfrac{1}{n+1}$. As consequences any projective curve C on X satisfies the inequality $2p_a(C) - 2 \geq \dfrac{1}{n+1} \int_C K_{X|C}$ where $p_a(C)$ is the* arithmetic genus of C. *In particular X does not contain neither rational curves nor curves of arithmetic genus 1.*

It is enough to mimic the proof of Corollary 5.1.4.

Finally, by [61, Proposition 9.8] we have the following theorem.

Theorem 5.2.9 (cfr. [69, Theorem 4 and Remark (iii)])**.** *Let X be a smooth complex projective variety such that K_X is ample and $GY(X) = 0$. Then X is uniformized by $\mathbb{B}^n$ (the unit ball of $\mathbb{C}^n$).*

Remark 5.2.10. By [70, Example 2.1.2.2], Corollary 5.2.4 can be viewed as a consequence of previous Theorem. $\Diamond$

Appendix A

1-H-nflat Higgs bundles over smooth complex projective varieties

The contents of this appendix are mainly based on paper [12].

We write this appendix because on one hand 1-H-nflat Higgs bundles satisfy the Conjecture 3. On another hand, the definition of these Higgs bundles involves Hermitian metric on the underlying vector bundle and connections on the relevant Hermitian vector bundle which are sensitive to Higgs field.

In this appendix, X is a smooth complex projective variety.

A.1 Definition and main properties

In [12], inspired by the work of De Cataldo [21] for ordinary vector bundles, a notion of numerical effectiveness for Higgs bundles was given in terms of bundle metrics. If $\mathfrak{E} = (E, \varphi)$ is a Higgs bundle, and h is a Hermitian metric on E, one defines the *Hitchin-Simpson connection* of the pair $(\mathfrak{E}, h)$ as

$$\mathcal{D}_{(h,\varphi)} = D_h + \varphi + \overline{\varphi}$$

where D_h is the Chern connection of the Hermitian bundle (E, h), and $\overline{\varphi}$ is the metric adjoint of φ defined as

$$h(s, \varphi(t)) = h\left(\overline{\varphi}(s), t\right)$$

for all sections s, t of E. The curvature $\mathcal{R}_{(h,\varphi)}$ of the Hitchin-Simpson connection defines a bilinear form on $TX \otimes E$, where TX is the tangent bundle to X, by letting

$$\widetilde{\mathcal{R}}_{(h,\varphi)}(u \otimes s, v \otimes t) = \frac{i}{2\pi} \left\langle h\left(\mathcal{R}^{(1,1)}_{(h,\varphi)}(s), t\right), u \otimes v \right\rangle .$$

where $\mathcal{R}^{(1,1)}_{(h,\varphi)}$ is the $(1,1)$-part of $\mathcal{R}_{(h,\varphi)}$, and $\langle\,,\rangle$ is the scalar product given by the Kähler form associated with the given polarization of X.

Definition A.1.1. A Higgs bundle $\mathfrak{E} = (E, \varphi)$ over X is said to be

a) 1-*H-nef* if for every $\xi > 0$ there exists a Hermitian metric h_ξ on E such that the bilinear form
$$\widetilde{\mathcal{R}}_{(h_\xi,\varphi)} + \xi\omega \otimes h_\xi$$
is semipositive definite on all sections of $TX \otimes E$ that, at every point x in their domain, define a rank one tensor in $(TX)_x \otimes E_x$ (here ω is the Kähler form given by the polarization of X);

b) 1-*H-nflat* if both $\mathfrak{E}$ and $\mathfrak{E}^\vee$ are 1-H-nef.

c) *Hermitian flat* if there exists a Hermitian metric on E such that the curvature $\mathcal{R}_{(h,\varphi)}$ of the Hitchin-Simpson connection of $(\mathfrak{E}, h)$ vanishes.

1-H-nef Higgs bundles satisfy properties analogous to those of H-nef Higgs bundles. These properties have been proved in [12]; here we list some of them for completeness.

Lemma A.1.2. *Let $\mathfrak{E}$ be a 1-H-nef Higgs bundle over X. The following statements hold.*

a) Let $f\colon Y \to X$ be a morphism of smooth complex projective varieties. Then $f^*\mathfrak{E}$ is 1-H-nef ([12, Proposition 3.3]).

b) Every quotient Higgs bundle of $\mathfrak{E}$ is 1-H-nef ([12, Proposition 3.7]).

c) Tensor products, exterior and symmetric powers of 1-H-nef Higgs bundles are 1-H-nef ([12, Propositions 3.4 and 3.5]).

d) An extension of 1-H-nef Higgs bundles is 1-H-nef ([12, Proposition 3.9]).

e) $\mathfrak{E}$ is H-nef ([12, Proposition 4.3]).

Remark A.1.3. Regarding the last statement, the opposite implication is known to hold for Higgs line bundles ([12, Remark 3.2.(ii)]), and for Higgs bundles over smooth projective curves ([12, Lemma 4.5]).

The best of our knowledge, it is unknown whether it holds in general, even if one sets to zero the Higgs field (compare [21, Section 3.1]). $\lozenge$

A.2 H-nflat vs. 1-H-nflat Higgs bundles

1-H-nflat Higgs bundles satisfy other properties which have been proved in [12, 9].

Lemma A.2.1. *Let $\mathfrak{E}$ be a* 1*-H-nflat Higgs bundle over X. The following statements hold.*

a) $\mathfrak{E}$ is semistable ([12, Theorem 3.11]).

b) The 1*-H-nflatness of $\mathfrak{E}$ is equivalent to the existence of a filtration in Higgs subbundles whose quotients are locally free, Hermitian flat Higgs bundles. As a consequence, all Chern classes of $\mathfrak{E}$ vanish. ([12, Theorem 3.16])*

In other words, the last statement proves that Conjecture 3 holds for 1-H-nflat Higgs bundles, in the smooth complex projective framework. Actually, using Theorem 4.3.5, we have proved the following theorem.

Theorem A.2.2 ([9, Theorem 5.2])**.** *The following conditions are equivalent.*

a) Every H-nflat Higgs bundle over X is 1*-H-nflat.*

b) Every H-nflat Higgs bundles over X has vanishing Chern classes.

Remark A.2.3 (cfr. [9, Remark 5.3])**.** If we set the Higgs field to zero in Theorem A.2.2, *i.e.* if we apply the Theorem to ordinary vector bundles, we obtain that the notions of 1-numerical flatness and numerical flatness coincide. $\Diamond$

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